Private Learning and Exit Decisions in Collaboration*

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Abstract

We study a continuous-time collaboration model in which two players participate in a joint project. At any time, players decide between exerting effort or exiting (irreversible) to secure the outside option's payoff. The arrival rate of a public success is proportional to players' efforts, depends on the binary state, and is positive in both. A player's effort is also an investment in private learning: They may privately learn that the state is bad, in which case exerting effort is inefficient.

We identify an equilibrium with three phases. Uninformed players consistently exert effort, while those informed about the project being bad stop exerting effort. In the first, no-exit phase, informed players do not exit. In the subsequent, gradual-exit phase, they exit at a finite rate. They exit immediately in the final phase. We find that, despite becoming more pessimistic over time, players may increase their effort in the no-exit phase. Moreover, endogenous deadlines can exist. The equilibrium exhibits two inefficiencies: delayed effort and information transmission. Increasing the outside option's payoff mitigates both inefficiencies and fosters collaboration.

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JEL Classifications: C73, D82, D83,

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1 Introduction

Most great projects require teams. Researchers join forces to investigate new topics, in the endeavour to achieve a breakthrough and to advance knowledge. Entrepreneurs form alliances to attract investors for innovative products or joint ventures, and start-ups collaborate to expand into emerging markets. A common feature that many of these team projects share is that initially the probability of a success is uncertain. By exerting (costly) effort team members can increase the probability of the arrival of a success. Active involvement in a project not only increases the chance of success but also provides collaborators with insights into the project's quality or potential. This raises intriguing questions: if collaborators privately discover negative information, (bad news) will they share it with their team members and quit? Or will they keep it to themselves, delaying disclosure to their partner? How will this affect the effort incentives of collaborators?

To study these questions, we consider a two-player team problem in which players can exert costly effort in order to increase the arrival rate of a success. A success arrives according to a Poisson process, is public, and rewards both team members with a lump-sum payoff. The arrival rate of a success is proportional to the sum of effort exerted by the players; it is higher in the good state than in the bad state.[1] Private learning about the quality of the project is captured by a new feature of the model: If the state is bad, a player who exerts effort may observe a private, fully-revealing signal (a *bad-state-revealing signal*). Such a signal is conclusive but private. Players have a positive outside option and can exit at any time, thereby quitting the project but securing the payoff of the outside option. Exits are public and irreversible. If a player decides to stay with the project, he chooses how much effort to exert. Efforts are unobservable.

Our goal in this paper is to understand the incentives and dynamics that arise from the novel features in the team problem that we consider: private learning and the option to exit. Notice that in our setting, efforts serve a dual purpose. On the one hand, they are a contribution to the joint task and increase the probability of a success. On the other hand, they are an investment in private learning. Exerting effort increases the probability of observing a private, bad-state-revealing signal, which renders the possibility to free-ride. A player who learns that the state is bad has two options: Stop exerting effort but remain with the project, hoping that the other player's effort will result in a success, or choose to exit and secure the positive payoff from the outside option.

Our model is an inconclusive good-news model: If no success arrives, then players become more pessimistic about the state being good and hence about the arrival rate of a success

[1] We allow for the success rate to be strictly positive in both states.

being high. However, the private, bad-state-revealing signal creates a countervailing effect. If a player does not observe a private signal, he becomes more optimistic about the state being good. Throughout the paper, we focus on the case in which the arrival rates are such that a single player gets more pessimistic about the state being good if no success or private signal arrives.[2]

The dynamic setting that we consider is complex and solving it requires to keep track of many beliefs, raising tractability issues. To tackle the problem, as a first step, we show that it suffices to keep track of two conditional beliefs – the probability that the state is good conditional on all players being uninformed and the probability that a player is uninformed conditional on the state being bad. By using this observation we are able to make the model tractable and to solve it generally. We identify all symmetric equilibria in this setting. This class of equilibria all share the following features: Any symmetric equilibrium consists of a subset of three phases, a *no-exit*, a *gradual-exit*, and an *immediate-exit* phase. The phases are named after the exit-behavior of an informed player. Depending on the parameter region, not all of these phases need to occur in equilibrium.[3] The game ends at a finite time, at which all players exit.

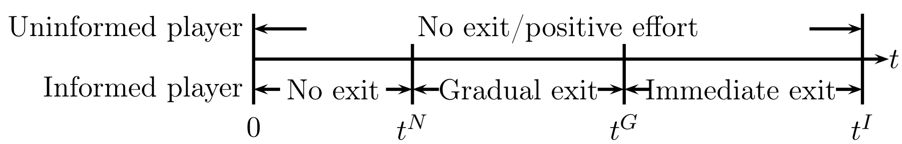


Figure 1: Structure of a three-phase equilibrium.

We now describe the equilibrium structure, which is illustrated in Figure 1, and provide some intuition for it. An *uninformed* player will always exert effort; whereas an *informed* player (who knows that the state is bad) never exerts effort in equilibrium.[4] However, initially his belief that his collaborator is still uninformed and hence is exerting effort is high. For sufficiently high prior beliefs, the expected payoff from staying with the project is higher than that of the outside option. Hence, an informed player stays with the project and free-rides on the expected effort from his opponent – play starts with a no-exit phase.

As time passes, players become more pessimistic about their opponent still being uninformed. Uninformed players must decide how to adjust their effort to the possible lack of effort from their potentially informed collaborator. For an informed player, it becomes more

[2]This seems to be a natural assumption in the applications that we have in mind.

[3]If at time zero, the players are sufficiently optimistic, a three-phase equilibrium exists which starts with a no-exit phase. For low priors, equilibria have only one – an immediate-exit – phase.

[4]We assume that exerting effort yields a positive net payoff in the good state and a negative net payoff in the bad state

likely that the other player is also informed and hence the project has reached a deadlock. The expected effort exerted by the other player decreases over time and it becomes less attractive for an informed player to remain with the project. At some time, $t^N \in [0, \infty)$, equilibrium play enters the gradual-exit phase. In the gradual-exit phase, an informed player is indifferent between staying and exiting, and exits at a finite rate. Uninformed players are never the first to exit on the equilibrium path. Observing that the opponent has not exited is good news and encourages uninformed players to keep exerting effort. However, arrival rates are such that an uninformed player becomes more pessimistic about the state being good as effort is put into the project, even if he was certain that his opponent is also uninformed. The increasing equilibrium exit rate in the gradual-exit phase counter-balances but cannot completely offset this effect. Consequently, over time, uninformed players become more pessimistic and hence decrease their effort level. At some time t^G, the game proceeds to the immediate-exit phase: any player who becomes informed exits immediately. The equilibrium effort is so low that the flow payoff from staying is strictly less than the payoff of the outside option, even when the opponent is uninformed and exerts effort with probability one. The immediate-exit phase lasts until a finite final-exit time t^I at which all players opt for the outside option. To see this, notice that players are not willing to put effort beyond the time at which beliefs drop to a level where the flow payoff of an uninformed player from staying equals that of the outside option.

There are two types of inefficiencies that arise in equilibrium. Since we study a team problem with moral hazard, players have an incentive to reduce and *delay effort*. The second inefficiency, *delayed information transmission*, arises from the new features in our model. A privately informed player has an incentive to delay his exit in order to free-ride on the other player's effort.

The identified equilibrium exhibits surprising novel dynamics. We find that in the no-exit phase, the equilibrium effort level may be increasing, even though players become more pessimistic over time. This is in stark contrast to the findings in the previous literature, in which effort levels typically decrease as players become more pessimistic.[5] Our model shares with this strand of literature the feature that players have the incentive to free-ride and delay effort. However, in our setting the possibility that the other player becomes informed and is hence not exerting effort reduces what is to be learned from the non-arrival of a success. Moreover, as we show an uninformed player may wish to compensate for the potential lack of effort of his informed team member, resulting in increasing effort in the no-exit phase. At the transition time between the no-exit and the gradual-exit phases, the uninformed player's effort level drops discontinuously. Intuitively, if an informed player exits with a positive

[5]See for example Bonatti and Hörner (2011), Keller et al. (2005), Keller and Rady (2015)

probability, an uninformed player has more incentive to postpone his effort in order to learn from the potential exit of his opponent. Hence, at the threshold time, effort levels must drop. In the gradual-exit and immediate-exit phase, effort levels are decreasing in equilibrium.

Another novel feature in our setting is that endogenous deadlines can prevail. The final-exit time is not uniquely determined but there is an interval of feasible final-exit times. Intuitively, any time at which the belief that the state is good has dropped to a level at which it is not optimal to continue working individually on the project if the partner leaves, is a feasible endogenous deadline. We show that for each of these final-exit times there exists a unique symmetric equilibrium, and that this is the full class of symmetric equilibria. All symmetric equilibria except the one with the longest duration of experimentation, display a *deadline effect*: In each of this equilibria with earlier final-exit time, there exists a jump-time at which efforts jump to one (full effort) and remain there until the end.[6]

We find that increasing the payoff of the outside option, and hence making it more attractive for a player to leave the project, encourages collaboration. More specifically, increasing the payoff of the outside option diminishes both inefficiencies, delayed effort and delayed information transmission. It may be surprising at first that making it more attractive for players to switch to the outside option—which seems detrimental to a partnership—will increase efficiency. However, within the partnership, players have an incentive to delay effort and to delay revealing their private information that the state is bad. Increasing the payoff of the outside option mitigates both of these inefficiencies and leads to a better alignment of players' incentives. For sufficiently high payoffs of the outside option, the equilibrium payoff equals the cooperative payoff. Uninformed players exert full effort, and informed players exit immediately.

Our paper contributes to the relatively small literature on private learning in experimentation models. Some recent, related papers are Akcigit and Liu (2016), Das and Klein (2020), and Bimpikis et al. (2018). Akcigit and Liu (2016) examine an innovation competition between two firms which decide whether to pursue a risky or a safe project. Only the first success of a project is rewarded. The risky project may be a success or a dead end, and firms may privately find out about dead ends. Since a firm benefits when its competitor works in a less rewarding direction, it never reveals dead-end findings – competition suppresses information sharing. By contrast, in our model information sharing may be delayed since an informed player has an incentive to free-ride on his opponent's effort. Das and Klein (2020) examine a situation in which two players can work on a risky project or a safe project, and only the first player who obtains a public success is rewarded. If the state is good, then in addition to a public success, the risky project may also generate private good news,

[6] A similar effect arises for the case of exogenous deadlines in Bonatti and Hörner (2011).

which encourages an informed player to stay with the risky option forever. Depending on the prior, players experiment either too much or not enough.[7] Bimpikis et al. (2018) study a strategic experimentation model in which players' actions are private. Information generated through experimentation is private, but can be credibly disclosed. They show that efficiency is improved if all players commit to share no information up to a time and to fully disclose all available information at that time. Unlike our paper, their setting involves information externalities only and no payoff externalities. Heidhues et al. (2015) study a strategic experimentation game with observable actions and private payoffs. They show that private payoffs can diminish the free-rider problem, and identify cases in which the cooperative solution can be supported as a perfect Bayesian equilibrium.

There are various paper that study asymmetric information in experimentation settings. Dong (2021) studies an experimentation setting in which two players each choose between allocating effort between a risky and safe alternative where the return of the risky alternative depends on an unknown state. A planner can choose whether to inform players symmetrically or asymmetrically, that is, only giving one player access to a noisy signal about the state prior to experimentation. Efforts and successes are observable, so players can learn from each other. Dong shows that under some conditions asymmetrically informing players lead to an encouragement effect and hence increases welfare as well as the total amount of learning. Cetemen (2021) considers team members with private information about a common productivity parameter. Effort is unobservable but players learn through a public noisy signal about total effort. Therefore, players exaggerate their efforts to signal their private information. As such, in this setting, asymmetric information leads to an encouragement effect that counteracts the free-riding incentives and may even result to inefficiently high effort levels. Campbell et al. (2014) study a partnership in which players work on a joint project with a deadline and have private information about the success of their efforts. In equilibrium, players initially reveal their information but exert inefficiently low effort. As the deadline draws closer, players hide their information about successes to encourage their partners to work more. They show that private information about successes benefits welfare, compared to the case in which successes are public.

Our model also ties into the literature on dynamic games with exit options. McAdams (2011) analyzes stochastic partnerships in which players can either stay with the current partner, or exit and get anonymously rematched. Players' actions are publicly observed; stage game payoffs vary stochastically and are common knowledge. McAdams (2011) shows

[7] Bergemann and Hege (2005) study agency problems regarding the timing of the termination of funding for R&D projects with uncertainty about the probability of success. They find that in equilibrium funding stops inefficiently early.

that performance inside the partnership decreases with the attractiveness of players' outside options. By contrast, in our model we obtain the opposite effect: increasing the attractiveness of the outside option encourages collaboration within the partnership. Moscarini and Squintani (2010) study an R&D, winner-takes-all setting, in which players hold private information about the arrival rate of success. Staying in the race is costly, but players can choose to publicly exit. Players learn from exit decisions of their competitors, and the equilibrium exhibits a strong "herding" effect. Even if players differ strongly in their costs and benefits, they may exit at almost the same time. This is attributed to the survivor's curse: at any time in the game, a player is more optimistic about the state and his opponent's information than if he knew that his opponent would exit in the next instant. Murto and Välimäki (2011) examine information aggregation in an exit game in which players are uncertain about their payoff types, and their types are correlated.[8] Good types should stay in the game whereas bad types are better off exiting. By staying with the project, good-type players may privately learn about their type. They show that information aggregates in randomly occurring exit waves.

More broadly, this paper is related to the literature on experimentation. (See, for instance, Bolton and Harris (1999), Keller et al. (2005), and Bonatti and Hörner (2011)). Our model is based on the collaboration model of Bonatti and Hörner (2011). They analyze moral hazard in teams, and show that the incentive to free-ride on other players' efforts leads to reduction of effort and procrastination. Their model is incorporated as a special case in our setting, in which the payoff of the outside option and the arrival rates of a private signal or a success in the bad state are all zero. As in Keller and Rady (2010), and the related bad-news model Keller and Rady (2015), we assume that the arrival rate of a success is positive in both states.

2 The model

Two players, $i \in \{1, 2\}$, are engaged in a joint project. Time is continuous with an infinite horizon. At each instant t, player i first decides whether to remain engaged in the project, or to exit and take his outside option. The outside option yields a flow payoff of $f > 0$.[9] A player's exit is public and irreversible. If player i stays with the project, he chooses his effort $a_i(t) \in [0, 1]$. The instantaneous cost to player i from exerting effort $a_i(t)$ is $ca_i(t)$. Player i's

[8] A related exit-game models with common values and private learning is studied in Rosenberg et al. (2007).

[9] The outside option can be interpreted as the expected payoff from starting a new project (if each player can be involved in at most one project), or as the opportunity cost associated with staying with the project that can be avoided by quitting.

effort choice is and remains his private information.

The project might generate one public success. The success yields a lump sum $h > 0$ to each of the players who are still engaged in the project. If a success arrives players have completed the project. Before completion players reap no benefits from the project. Once players complete the project, they take their outside options immediately.

Arrival of a success. The probability of completing the project depends on players' efforts, and on an unknown binary state which is either good g or bad b. A success arrives according to a Poisson process; the arrival rate is proportional to the joint effort. If players' efforts are $(a_1(t), a_2(t))$ at time t, then the instantaneous probability of success is $\lambda_g(a_1(t) + a_2(t))$ if the state is good, and $\lambda_b(a_1(t)+a_2(t))$ if the state is bad, with $\lambda_g > \lambda_b \geqslant 0$. In most of the paper, we assume that the probability of a success is strictly positive in both states (i.e., $\lambda_b > 0$).[10]

We assume that exerting effort yields a positive net payoff in the good state and a negative net payoff in the bad state, $h\lambda_b < c < h\lambda_g$. Both players share a common prior belief $p^g(0) \in (0, 1)$ that the state is good. We say that efforts are *individually productive* at time t, if the payoff rate from effort is higher than c:

$$p^g(t) \geqslant \frac{c - \lambda_b h}{(\lambda_g - \lambda_b)h}.$$

Throughout the paper, we assume that the prior belief is high enough, such that at time $t = 0$ effort are individually productive, that is, an individual player would want to take up the project and exert effort.

Private learning. If a player exerts effort, he may privately learn that the state is bad. If player i exerts effort $a_i(t)$ at time t, then in the bad state, a private signal arrives with instantaneous probability equal to $\beta a_i(t)$, with $\beta > 0$. Hence, exerting more effort increases the likelihood of receiving a signal in the bad state. No such signal arrives in the good state. The arrival of a signal reveals that the state is bad, thus, we also refer to the signal as a *bad-state-revealing* signal. We say that a player is *informed*, if he has obtained such a signal and knows that the state is bad, while an *uninformed* player is uncertain about the state.

Payoffs. Players discount future benefits and costs at a common rate r. If player i exerts effort $(a_i(t))_{t\geqslant 0}$, exits at time $\tau \leqslant \infty$, and a success occurs at time t_s, then player i's

[10]Technically, our analysis includes the case $\lambda_b = 0$, in which an informed player would always want to exit immediately and secure the payoff of the outside option.

normalized discounted payoff is

$$-r\int_0^\tau e^{-rs}ca_i\left(s\right)\mathrm{d}s+e^{-r\tau}f+\mathbb{1}_{t_s\leqslant\tau}\cdot re^{-rt_s}h,$$

where $\mathbb{1}_{t_s\leqslant\tau}$ is one if a success occurs before player i exits, and zero otherwise. If a success occurs before player i exits, he shall take the outside option immediately after the success because the project generates at most one success. We assume so for the remainder of the paper. A player's objective is to maximize his expected payoff by choosing his effort levels and his exit time.

Strategies and solution concept. In our model, we have to keep track of public and private histories. At any time t, the public history captures whether and when some player has exited or a success has arrived. Player i's private history consists of his past efforts, and whether and when he has observed a private signal. For player i, the history at time t consists of both the public and his private history, and is denoted by $h_{i,t}$.

In order to circumvent modeling issues that arise in continuous time models, we formulate the game as one with a random number of stages, and partition the set of histories into subsets of stage game histories.[11] Figure 2 illustrates the stages and transitions from the perspective of player i. All stages are conditional on no success having arrived yet. Throughout, we use j to denote player i's opponent.

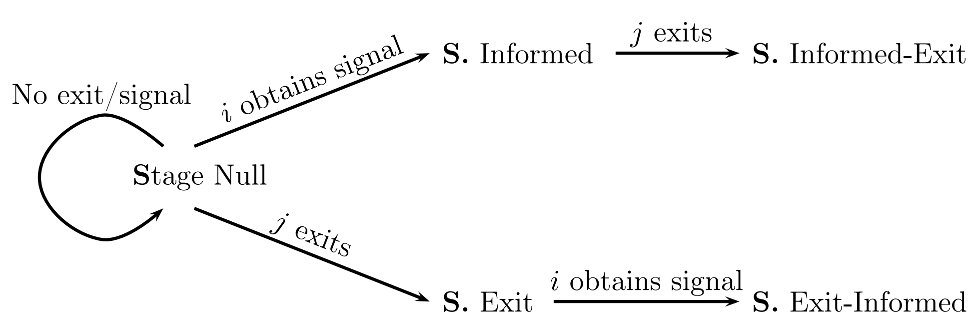


Figure 2: Stages of the game for player i.

Every history of player i in which he has not exited yet, falls into exactly one stage $m\in M:=\{\text{Null},\text{Informed},\text{Exit},\text{Informed-Exit},\text{Exit-Informed}\}$. Let $\mathcal{H}_i^m$ be the set of stage

[11]The problem we allude to is the following: If a player receives a private signal or observes an exit, his information set changes. Hence, a player may want to react immediately to a signal or to another player's exit decision. It is well known that this may create issues regarding the timing of events in continuous time models. To address this problem, we adopt an approach similar to the one in Murto and Välimäki (2013) and Akcigit and Liu (2016).

m histories for player i. The set of histories for player i is $\mathcal{H}_i = \cup_m \mathcal{H}_i^m$. For any m, a stage m strategy for player i includes two measurable functions, which specify the effort level and the exit rate conditional on staying in stage m:

$$a_i^m : \mathcal{H}_i^m \to [0,1]\,, \ \text{ and } \ d_i^m : \mathcal{H}_i^m \to [0,\infty]\,.$$

Here, $a_i^m(h_{i,t}^m)$ and $d_i^m(h_{i,t}^m)$ are the effort level and the exit rate given history $h_{i,t}^m \in \mathcal{H}_i^m$.[12] A strategy of player i is given by $\{(a_i^m, d_i^m)\}_{m\in M}$, consisting of a stage m strategy of player i for every $m \in M$.

The equilibrium concept is perfect Bayesian equilibrium (PBE). We focus on symmetric equilibria. Any strategy profile induces the beliefs of the players. A strategy profile $\{\{(a_i^m, d_i^m)\}_{m\in M}\}_{i\in\{1,2\}}$ and a belief profile is a PBE if (i) beliefs are updated by Bayes' rule whenever possible, and (ii) for each i and all $h_{i,t}$, the continuation of $\{(a_i^m, d_i^m)\}_{m\in M}$ after $h_{i,t}$ is a best response to player j's strategy. Within each stage we focus on Markov strategies that depend only on a player's beliefs.

Notice that player i faces the single-player problem after j exits. This problem is solved in section 3. Hence, we only need to identify the effort and exit decision in stage Null and stage Informed. Stage Null corresponds to player i being uninformed, and stage Informed corresponds to player i being informed.[13] Lemma A.1 in the appendix shows that in any equilibrium an informed player exerts no effort, so we only need to specify his exit decision. We also show in Lemma A.2 that if an informed player prefers to stay, then an uninformed player strictly prefers to stay. Therefore, we only need to specify an uninformed player's effort choice and a final time at which both uninformed players exit. By a slight abuse of notation, we will use $a_i(t)$ for an uninformed player i's effort level, and $d_i(t)$ for an informed player i's exit rate.

3 Cooperative and single-player solution

As a benchmark, we analyze the cooperative problem in which $n \in \mathbb{N}$ players choose a strategy profile to maximize their average expected payoff. It is without loss to focus on symmetric strategy profiles. The case $n = 1$ corresponds to a single player's optimal strategy.

Working cooperatively, a player internalizes the benefit of his effort to his teammates. A success generates a payoff of h to every player. Hence, given the belief $p^g(t)$ of state g, the

[12] In section 4, we explain how an exit-rate strategy can be interpreted as choosing an exit time according to a certain distribution after observing a private signal.

[13] Transitions induced by private signals lead to private stages. Hence, stage Null and stage Informed of player i are indistinguishable for player j, and are private stages for player i.

payoff rate generated by an individual player's effort is

$$nh \cdot (p^g(t)\lambda_g + (1 - p^g(t))\lambda_b) - c.$$

If this payoff rate is higher than that of the outside option, f, then all players exert full effort. Otherwise, all players should exit and take their outside option.

For a sufficiently large team, the payoff rate from effort is higher than that of the outside option, even when the state is bad, $nh\lambda_b - c \geqslant f$. In this case, it is optimal for all players to exert full effort until they complete the project.

For smaller teams exerting effort in the bad state is inefficient (too costly), $nh\lambda_b - c < f$. Hence all players should take the outside option if they learn that the state is bad. If a player receives a signal, he should exit immediately and his teammates should follow suit. Thus, if no player has exited, this means that no player has yet observed a bad-state-revealing signal. Consequently, under the cooperative solution either all players are still involved in the project and exert full effort, or all have exited.

As a result, in the cooperative game, players always share a common belief that the state is good. At any time t, given the belief $p^g(t)$, if players exert efforts $(a_1, \ldots, a_n)$ over the interval $[t,\, t + \mathrm{d}t)$, the posterior belief conditional on no success and no exit is given by

$$p^g(t + \,\mathrm{d}t) = \frac{p^g(t)e^{-\lambda_g\left(\sum_{i=1}^n a_i\right)\mathrm{d}t}}{p^g(t)e^{-\lambda_g\left(\sum_{i=1}^n a_i\right)\mathrm{d}t} + (1 - p^g(t))e^{-(\lambda_b+\beta)\left(\sum_{i=1}^n a_i\right)\mathrm{d}t}}. \tag{1}$$

The non-arrival of a success makes players more pessimistic about the state being good. However, the non-arrival of a signal counteracts this affect and makes players more optimistic. In our analysis, we focus on the case, $\beta \leq \lambda_g - \lambda_b$ in which private learning is *slow*. In this case, if no success or signal arrives, players become (weakly) more pessimistic that the state is g. If $\beta < \lambda_g - \lambda_b$, then the lack of a signal does not fully compensate for the lack of a success, and p^g is strictly decreasing. If $\beta = \lambda_g - \lambda_b$ the lack of a signal exactly offsets the lack of a success, and p^g stays constant as long as no success or signal arrives. We call this special case the *stationary case*.

The following proposition summarizes the discussion on the cooperative solution. The proof is in the appendix.

Proposition 3.1 (Cooperative Solution)**.**
Consider the cooperative problem with n players. Then

(i) if $nh\lambda_b - c \geqslant f$, players exert full effort until a success occurs and then exit.

(ii) if $nh\lambda_b - c < f$, a player exits when a success occurs, he observes a private signal, or another player exits. Without a success, a private signal or an exit, the belief of state

g evolves according to (1). *Players exert full effort if this belief is above the following threshold and exit if below:*

$$p_n^* := \frac{c + f - nh\lambda_b}{nh(\lambda_g - \lambda_b)}.$$

Under the cooperative solutions, all players stay with the project and exert full effort if the belief of state g is above p_n^*. Notice that, the cooperative threshold p_n^* does not depend on β. The arrival rate of a signal affects the motion of the belief (1) but not the belief threshold at which it is optimal to stop exerting effort. Moreover, the threshold p_n^* decreases in n. Hence, a larger team gives up the project at a lower threshold belief than a smaller team.

In the cooperative solution, no player procrastinates in putting forth effort. When a player observes a bad-state-revealing signal and $nh\lambda_b - c < f$, he reveals this information by exiting. His teammates learn from this exit that the state is bad, and follow suit. Therefore, there is no delay in information transmission. We will show later that neither of these observations—no delay in exerting effort or in information transmission—holds in the noncooperative game.

3.1 The role of the outside option

Before we move on to the equilibrium analysis, we discuss how the outside option affects a player's behavior. Specifically, we want to highlight two distinct roles of the outside option and how they have very different effects on players' behavior. The intuition gained through this discussion will help to understand the intuition behind one of our main results. As we will show, a player's effort level may increase in the initial phase of the equilibrium even though players become more pessimistic over time (see Proposition 7.1).

To illustrate the different roles of the outside option, we consider the single-player problem of our model, in which the player can exit whenever he likes. We compare this to the case in which a player can exit only after a success. We shall emphasize that in our equilibrium analysis, we will always assume that players can exit whenever they like. However, there will be an equilibrium phase in which players exit only after a success. For the effort choice of a player in this phase, the situation is as if he could exit only after a success. When providing intuition for the effort evolution in that equilibrium phase, we will refer back to the following discussion of the two environments.

Consider a single player who can exit whenever he likes. Given the belief $p^g(t)$ of state g, the player stays with the project and exerts full effort if the payoff rate from effort is higher than that of the outside option:

$$(p^g(t)\lambda_g + (1 - p^g(t))\lambda_b)h - c \geqslant f.$$

Otherwise, he takes the outside option. A more attractive outside option—higher f—leads to a higher threshold belief at which the player optimally exits. Hence, for a player who can exit at any time, a higher outside option diminishes a player's incentive to stay with the project and exert effort.

Now consider a single player who cannot exit unless a success occurs. Given the belief $p^g(t)$ of state g, he is willing to exert effort if and only if

$$(p^g(t)\lambda_g + (1 - p^g(t))\lambda_b)\left(h + \frac{f}{r}\right) - c \geqslant 0. \tag{2}$$

We refer to the left-hand side term as *the markup of effort* given the belief $p^g(t)$. It captures the instantaneous net value from exerting effort for a player who won't exit without a success. To see this, notice that f/r is the discounted sum of the flow payoffs from the outside option. For a player who is "trapped" in the project, the situation is as if the value of a success is $(h + f/r)$ instead of h. Compared with the first environment, the role of the outside option changes completely. A higher f leads to a stronger incentive to exert effort. Moreover, f/r decreases in r, so a more patient player has a stronger incentive to exert effort.

Notice that, even with our assumption that $\lambda_b h < c$, the markup of effort conditional on the state being bad can be positive, as long as f/r is high enough. The outside option adds to the value of a success and hence to the incentive to exert effort. In section 7, we will see that this influences the evolution of the effort level of an uninformed player in the equilibrium phase in which players exit only after a success.[14]

4 Equilibrium Effort Level and Exit Rate

In equilibrium, beliefs and the strategy of the other player determine a player's optimal effort level and exit rate, which in turn determine the motion of equilibrium beliefs. We start off by discussing optimal effort levels and exit rates for a given set of beliefs. The evolution of beliefs is discussed in section 5. Putting these insights together we then characterize equilibria in section 6. Recall that it is never optimal for informed players to exert effort or uninformed players to exit first (see Appendix A). Hence, we restrict attention to determining the effort level $a_i(t)$ of uninformed players and the exit rate $d_i(t)$ of informed players.

[14] The property that in our model players choose both, whether or not to exit and how much effort to exert, is the reason why both terms $(p^g(t)\lambda_g + (1 - p^g(t))\lambda_b)h - c - f$ and the markup of effort $(p^g(t)\lambda_g + (1 - p^g(t))\lambda_b)(h + f/r) - c$ are relevant for our analysis. In models without an exit option, like Bonatti and Hörner (2011), the distinction between those two terms is not relevant and the payoff of the outside option can be normalized to zero.

Simplifying the problem. Throughout the game, a player needs to keep track of the probabilities that (i) the state is good, (ii) the state is bad and his opponent is *informed*, and (iii) the state is bad and his opponent is *uninformed*. We denote these beliefs at time t by $p^g(t)$, $p^{bi}(t)$, $p^{bu}(t)$, respectively.[15] At any time t, these beliefs sum up to one, and hence there are only two degrees of freedom. Therefore, it suffices to keep track of only two beliefs. It is sometimes easier to work with the following two conditional beliefs:

$$q^u(t) := \frac{p^{bu}(t)}{p^{bi}(t) + p^{bu}(t)}, \quad q^g(t) := \frac{p^g(t)}{p^g(t) + p^{bu}(t)}. \tag{3}$$

Here, q^u is the probability that a player's opponent is uninformed conditional on the state being bad, and q^g is the probability that the state is good conditional on both players being uninformed.[16] We will use both (p^g, p^{bi}, p^{bu}) and (q^u, q^g) in our analysis.

Both uninformed and informed players assign the same probability to the event that their opponent is uninformed conditional on the state being bad, $q^u(t)$. An informed player learns that the state is bad, i.e., $q^g = p^g = 0$. He only keeps track of the probability that his opponent is uninformed.

As we will see in section 6, equilibrium play may go through different phases as beliefs evolve. For our analysis, we use that the exit decision of an informed player can be divided into two cases: exit with certainty, or exit at a finite – possibly zero – rate. We discuss these two cases separately. Which of these options is optimal for an informed player depends on the player's beliefs as well as the effort level of the other player. If an informed player is optimistic enough about his opponent being uninformed and exerting sufficiently high effort, then it is optimal for the informed player to stay with the project and delay his exit. He can either stay with certainty (*no-exit phase*) or gradually exit with a certain probability (*gradual-exit phase*), which will lead to a distribution over exit-times.[17] If an informed player is more pessimistic that his opponent is still uninformed, or if efforts of uninformed players

[15] The superscript "bi" refers to a player's belief in the event that the state is bad and his opponent is informed; "bu" refers to the event that the state is bad and his opponent is uninformed. All of these beliefs are conditional on no success having arrived yet. We only need to keep track of these beliefs, since it is optimal for all players to exit immediately after a success.

[16] Note that $q^g(t) \geqslant p^g(t)$: an uninformed player would be more optimistic that the state is good if he knew that his opponent were uninformed. In the stationary case $\beta = \lambda_g - \lambda_b$, the belief $q^g(t)$ stays constant. It is always equal to the prior $p^g(0)$. As a result, there is only one degree of freedom for the beliefs $p^g(t)$, $p^{bi}(t)$, $p^{bu}(t)$.

[17] The exit-rate $d_j(t)$ can be interpreted as choosing an exit time according to a certain distribution. In particular, a player who becomes informed at t^i chooses to exit at $t \geqslant t^i$ according to the distribution $1 - e^{-\int_{t^i}^{t} d_j(s)\mathrm{d}s}$. Recall here, that $d_j = 0$ during the no-exit phase, that is a player who becomes informed during a no-exit phase, will only start exiting once the game transitions to a gradual- or immediate-exit phase.

are low, then it is optimal for an informed player to exit (*immedite-exit phase*).

Phases in which informed players exit at a finite rate. For an informed player to delay his exit, he must (weakly) prefer staying over exiting. At such a time t, the flow payoff from the project must be (weakly) higher than that of the outside option:

$$\lambda_b h q^u(t) a_j(t) \geqslant f. \tag{4}$$

In the gradual-exit phase, informed players must be indifferent between staying and exiting: (4) must hold with equality.

An uninformed player must choose an optimal effort level given his beliefs in equilibrium, which implies that he must have no incentive to either postpone or advance effort. In equilibrium, effort levels are such that the other player is indifferent across time. Using this property and the exit-behavior of informed players determines the effort level and exit rate in any no-exit or gradual-exit phase, as characterized in Lemma 4.1 below.

For ease of exposition, we define the following arrival rates as functions of (p^g, p^{bi}, p^{bu}):

$$\begin{aligned} \lambda^s(p^g) &:= p^g \lambda_g + (1-p^g)\lambda_b \\ \lambda^{s,i}(p^g) &:= \lambda^s(p^g) + (1-p^g)\beta \\ \lambda^u(p^g, p^{bu}) &:= p^g \lambda_g + p^{bu}\lambda_b. \end{aligned} \tag{5}$$

Here, $\lambda^s(p^g)$ is the arrival rate of a success, and $\lambda^{s,i}(p^g)$ is the arrival rate of a success or a signal, generated by an uninformed player's own effort. Moreover, $\lambda^u(p^g, p^{bu})$ is the arrival rate of a success generated by a player's opponent's effort – recall that the opponent exerts effort only if he is uninformed.

Lemma 4.1 (Effort level in a no-exit or gradual-exit phase)**.** *In any phase in which informed players delay their exit, $d_j \in [0, \infty)$, the equilibrium effort level and exit rate satisfy:*

$$a_j = \min\left\{ \frac{\lambda^s(p^g)\,(hr+f) - cr - d_j \cdot (1-p^g-p^{bu})(c-h\lambda_b)}{\lambda^u(p^g,p^{bu})c},\ 1 \right\}. \tag{6}$$

Moreover,

1. *in a no-exit phase, $d_j^N = 0$,*

2. *in a gradual-exit phase, $d_j^G > 0$, and the effort level and exit rate are given by a solution to the system* (6) *and* $a_j = \frac{f}{\lambda_b h q^u}$.

Effort levels must be such that an uninformed opponent wants to exert the same effort level and has no incentive to advance or postpone efforts. Not surprisingly, the higher the exit rate of informed players, the lower the effort an uninformed player is willing to put. We now present heuristic arguments to provide some intuition for how these effort levels are obtained. The formal proof is relegated to the appendix.

In equilibrium, if an uninformed player's effort is interior, then he has no incentive to either postpone or advance effort. Consider time t, and suppose that an uninformed player i exerts effort a_i over the interval $[t, t + \mathrm{d}t)$ (*today*) and effort a_i' over the interval $[t + \mathrm{d}t, t + 2\,\mathrm{d}t)$ (*tomorrow*). Now, consider the effect if player i decreases his effort today by ε and increases his effort tomorrow by the same amount. Conditional on reaching $t + 2\,\mathrm{d}t$ without a success or a signal, this change has no impact on the beliefs at $t + 2\,\mathrm{d}t$, and hence no impact on his continuation payoff.

Exerting this ε effort today increases the probability of the arrival of a success or a signal, at rate $\lambda^{s,i}(p^g)\varepsilon$. In either event, player i saves the cost of the planned effort tomorrow, which is ca_i'. If instead player i postpones this ε effort until tomorrow, then there is a chance that this ε effort will not be carried out. This is the case if a success or a signal arrives today, the probability of which is $\lambda^{s,i}(p^g)a_i + \lambda^u(p^g, p^{bu})a_j$. The cost saved is $c\varepsilon$. If player i postpones effort ε, then there is a chance that his opponent exits today. The instantaneous probability of this event is $p^{bi}d_j$. If player j exits, then player i saves the cost of this postponed effort $c\varepsilon$, but he also forgoes the chance of an instantaneous success, which would yield an expected payoff $h\lambda_b\varepsilon$. Lastly, given that players are impatient, there is another cost of postponing. The markup of effort $\left(\lambda^s(p^g)\left(h + \frac{f}{r}\right) - c\right) \cdot \varepsilon$ is delayed at a cost. Postponing effort until tomorrow is profitable if and only if:

$$\underbrace{\left(\lambda^{s,i}(p^g)a_i + \lambda^u(p^g, p^{bu})a_j\right) c}_{\substack{\text{saved cost upon arrival} \\ \text{of a success or a signal today}}} - \underbrace{r\left(\lambda^s(p^g)\left(h + \frac{f}{r}\right) - c\right)}_{\substack{\text{cost of delayed} \\ \text{markup of effort}}} + \underbrace{p^{bi}d_j(c - h\lambda_b)}_{\substack{\text{learning from} \\ \text{opponent's exit}}} \geqslant \underbrace{\lambda^{s,i}(p^g) \cdot ca_i'.}_{\substack{\text{benefit of} \\ \text{advancing effort}}} \tag{7}$$

The second term is zero in the no-exit phase. From (7) and the continuity of effort, it follows that the equilibrium effort – if interior – must satisfy

$$a_j = \frac{\lambda^s(p^g)\,(hr + f) - cr - d_j \cdot (1 - p^g - p^{bu})(c - h\lambda_b)}{\lambda^u(p^g, p^{bu})c}. \tag{8}$$

If the right-hand side of (8) is greater than one, then for any $a_j \leq 1$, player i would like to advance effort, and it follows that equilibrium effort levels must be one.

In the no-exit phase, if equilibrium efforts are interior, they are given by (8) with $d_j = 0$. In the gradual-exit phase informed players exit at a finite rate, $d_j > 0$: An informed player must be indifferent between exiting and staying. His flow payoff from staying is equal to that of the outside option, (4) holds with equality.[18] Combining (4) and (8) yields the equilibrium effort level and exit rate during phases in which informed players delay their exit.

Immediate-exit phase. In an immediate-exit phase, an informed player exits immediately. His opponent optimally follows, since an exit reveals that the state is bad. The situation is as if private signals were public. On the equilibrium path, both players – if they stay – are uninformed.

An immediate-exit phase can only exist if an informed player wants to exit immediately. Hence, it must be satisfied that $\lambda_b h \cdot a_j \leqslant f$. Given the known behavior of informed players in the immediate-exit phase, we can characterize an uninformed player's effort level. Again we use that the equilibrium effort level – if interior – is such that players have no incentive to postpone or advance effort.

Lemma 4.2 (Effort level in an immediate-exit phase)**.** *If an informed player exits immediately, then an uninformed player's equilibrium effort is given by:*

$$a_j = \min\left\{\frac{r(h\lambda^s(p^g) - c)}{c\lambda^{s,i}(p^g)} + \frac{f}{c},\ 1\right\}. \tag{9}$$

If an uninformed player exerts interior effort, then to determine equilibrium efforts we again compare the effect of shifting ε effort from today to tomorrow. Exerting this ε effort today increases the probability of the arrival of a success or a signal at rate $\lambda^{s,i}(p^g)\varepsilon$. In this case, an uninformed player i saves the cost of the planned effort tomorrow, which is ca_i'. If instead player i postpones this ε effort until tomorrow, then this ε effort is saved if a success or a signal arrives or player j exits today. The probability of this event is $\lambda^{s,i}(p^g)(a_i + a_j)$ and the cost saved is $c\varepsilon$. Lastly, there is also a cost of postponing due to impatience. This cost includes the cost of a delayed success and the cost of a delayed signal. It follows that postponing effort is profitable if and only if

$$\underbrace{\lambda^{s,i}(p^g)(a_i + a_j)c}_{\substack{\text{saved cost upon arrival}\\ \text{of a success/signal/exit today}}} \underbrace{- r\left(\lambda^s(p^g)\left(h + \frac{f}{r}\right) + (1-p^g)\beta\frac{f}{r} - c\right)}_{\substack{\text{cost of delayed success}\\ \text{and delayed signal}}} \geqslant \underbrace{\lambda^{s,i}(p^g)ca_i'.}_{\substack{\text{benefit of}\\ \text{advancing effort}}} \tag{10}$$

[18] Notice that this implies that there cannot be a time interval with non-empty interior in the gradual-exit phase in which efforts are one, except if the belief q^u is constant in this interval.

There are three differences between (10) and (7). First, the opponent is informed with probability zero, so $p^{bi} = 0$ (or equivalently $p^{bu} = 1 - p^g$). Second, whenever the opponent obtains a signal, he reveals the signal immediately by exiting. The postponed effort is saved in that event as well. These two differences explain why we replace $\lambda^u(p^g, p^{bu})$ with $\lambda^{s,i}(p^g)$. Third, player i himself exits immediately if he obtains a signal. Hence, the delayed arrival of a signal from postponing effort leads to the delayed consumption of the outside option.

In equilibrium, the effort level and exit rate determine the motion of beliefs and vice versa. For a given belief tupel $(q^g,\ q^u)$, a specific phase can only exist if the effort levels and exit rates identified in section 4 are non-negative and the specific exit-behavior for the given phase is optimal for informed players. In the appendix, B.3, we provide formal details and identify the sets of belief tuples (q^u, q^g) such that (i) a no-exit phase, (ii) a gradual-exit phase, and (iii) an immediate-exit phase can exist.

5 Equilibrium Belief Evolution and Phase Transitions

To identify equilibria, the belief realizations at any point in time are relevant, but also along which path these beliefs evolve. At time $t = 0$, the belief that the other player is uninformed is one, $q^u = 1$; the belief q^g equals the prior $q^g = p^g(0)$. The following lemma characterizes the evolution of these beliefs in each of the phases.

Lemma 5.1 (Belief evolution in equilibrium phases)**.** *In equilibrium,*

(i) the belief q^g is decreasing in all phases,

(ii) the belief q^u is decreasing in the no-exit phase, and constant $q^u \equiv 1$ in the immediate exit phase. There exists a continuous, decreasing function ψ such that, in the gradual-exit phase, q^u is increasing if and only if[19]

$$q^g \geq \psi(q^u).$$

Since $\lambda_g - \lambda_b \geq \beta$, the good news from the non-arrival of a bad-state revealing signal does not completely offset the bad news from the non-arrival of a success. Hence, q^g is (weakly) decreasing in all phases: Players get more pessimistic that the state is good conditional on both players being uninformed. The evolution of q^u is not as straightforward: In the no-exit phase, informed players do not exit. Therefore, players become more pessimistic that the other player is still uninformed conditional on the state being bad – q^u decreases over time.

[19] A formal definition of the function ψ is given by (28) in the appendix.

In the gradual-exit phase, the belief q^u may be increasing or decreasing. On the one hand, if the state is bad, any uninformed player may receive a private signal, which drives q^u down. On the other hand, informed players exit at a finite rate, which drives q^u up. For any tuple of beliefs (q^g, q^u), we can determine the corresponding equilibrium effort level and exit-rate candidates a_j and d_j. We can then identify whether q^u would be increasing or decreasing on the equilibrium path. For given parameters we can derive the function ψ such that q^u is increasing if and only if

$$q^g \geq \psi(q^u).$$

This is illustrated by the solid curve in Figure 3. If beliefs (q^u, q^g) are in the area above this

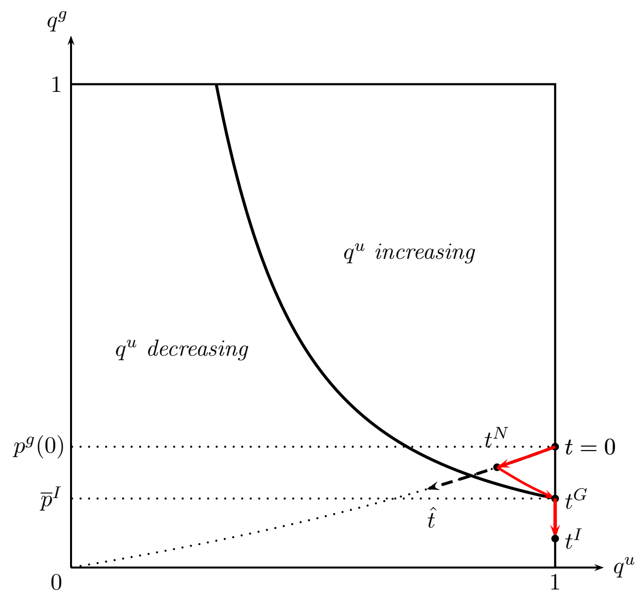


Figure 3: Motion of equilibrium beliefs.

curve, then the equilibrium exit-rate of informed players would be sufficiently high such that the belief q^u is increasing in the gradual-exit phase. Otherwise q^u is decreasing. We will show that, in equilibrium, beliefs in the gradual-exit phase stay above $\psi(\cdot)$ such that q^u and the exit rate d_j increase in time. Moreover, there exists some time t^G, such that, for $t \to t^G$, the belief q^u converges to one. In the immediate-exit phase, all players that are still with the project are uninformed, $q^u \equiv 1$ throughout.

Phase transitions. Finally, we need to identify, which phase transitions are feasible in equilibrium.

Lemma 5.2 (Feasible phase transitions). *In equilibrium,* *(i) the only feasible phase transitions are from the no-exit to the gradual-exit phase, and from the gradual-exit to the immediate-exit phase.*

(ii) The no-exit phase cannot last forever. Uninformed players only exit from the immediate-exit phase. If $\lambda_g - \lambda_b < \beta$, then the game ends at a finite time.

Let us provide some intuition for this result, the formal proof is relegated to the appendix. First, notice that the game cannot proceed from the no-exit phase to an immediate-exit phase in which an informed player exits for sure. To see this, notice that in the no-exit phase, q^u is strictly decreasing. Hence, after a non-trivial no-exit phase, $q^u < 1$ – the probability that one's opponent is informed is strictly positive. If the game would then transition from the no-exit phase to an immediate-exit phase, say at time $\hat{t}$, then right before this transition time an uninformed player would have no incentive to exert effort during $[\hat{t} - \mathrm{d}t, \hat{t})$, since he would expect to learn from his opponents potential exit at $\hat{t}$. But then an informed player has no incentive to stay during $[\hat{t} - \mathrm{d}t, \hat{t})$, which shows that the game cannot transition from the no-exit to the immediate-exit phase, it must first transition to a gradual-exit phase. We furthermore show that the no-exit – and if $\lambda_g - \lambda_b < \beta$ also the gradual-exit phase – cannot last forever. Play must eventually transition to the immediate-exit phase.

Finally, to see that the game generically ends at a finite time, recall that, in the immediate-exit phase all players are uninformed, and become more pessimistic over time – strictly, if $\lambda_g - \lambda_b < \beta$. The immediate-exit phase can only last until the belief that the state is good drops to the level such that the flow payoff from the project is equal to the outside option:

$$a_i(\lambda^s(p^g(\bar{t}^I))h - c) + a_j\lambda^s(p^g(\bar{t}^I))h = 2h\lambda^s(p^g(\bar{t}^I)) \cdot a_j - c = f.$$

Uninformed players are not willing to exert effort beyond this time $\bar{t}^I$, and hence all players exit. The beliefs of uninformed players at this time of exit depends on whether players are exerting full or interior effort at the final-exit time $\bar{t}^I$.

Corollary 5.1 (Beliefs at final-exit times)**.** *If $\lambda_g - \lambda_b < \beta$, then there exists a time $\bar{t}^I$, at which the belief $q^g(\bar{t}^I)$ drops to a level, such that uninformed players will not exert effort beyond this time. For $f < \lambda_b h$, this belief is $\frac{c - h\lambda_b}{h(\lambda_g - \lambda_b)}$; for $f \geqslant \lambda_b h$ this belief is* $\max\{\frac{c-h\lambda_b}{h(\lambda_g-\lambda_b)},\ p_2^*\}$.

The latter part of this result captures the case, in which uninformed players exert full effort until the final-exit time; they exit when $2h\lambda^s(p^g) - c = f$. If efforts are interior at the final exit time, then at this time for uninformed players the marginal benefit from effort, $h\lambda^s(p^g)$, is equal to the marginal cost c. Uninformed players are indifferent between all effort levels and, according to (9), choose the effort level at $a_j = f/c$ if interior. But then, for an uninformed player i – who benefits from his opponent's effort – this effort level generates a flow payoff at the same level as the outside option, that is, $a_j \cdot h\lambda^s(p^g) = f/c \cdot c = f$. All players exit at this time, $\bar{t}^I$, and take the outside option.

An exemplary equilibrium belief path is illustrated as the red line in Figure 3.

6 Equilibrium with Longest Experimentation

Recall that the equilibrium phases are classified by the exit-behavior of an informed player. At any time t an informed player either never exits (no-exit phase), exits at a finite rate (gradual-exit phase), or exits for sure (immediate-exit phase). Depending on the parameter region, not all of these phases need to occur in equilibrium.[20] From the analysis in section 4 we know that for given q^u, q^g, effort level and exit rate are pinned down within each phase. From section 5, we know how beliefs evolve and which phase transitions are feasible.

In order for an equilibrium with all three-phase (a *three-phase equilibrium*) to exist, the prior belief must be high enough such that there is an initial no-exit phase, that is, an informed player must have an incentive to stay with the project. The highest flow payoff that an informed player can obtain from staying with the project is $\lambda_b h$. This is the payoff rate in the case that the opponent exerts full effort with probability one. If the outside option f is higher than $\lambda_b h$, then it is a dominant strategy for an informed player to take the outside option immediately after he obtains a private signal. Given that equilibrium efforts are often interior an informed player may want to delay his exit, even if $\lambda_b h > f$. Using (4) and (6), we obtain that initially an informed player may want to delay his exit if

$$f \leq \lambda_b h \;\; \textit{and} \;\; p^g(0) = q^g(0) \geqslant F^{-1}(1) := \frac{(c - h\lambda_b)(\lambda_b(f + hr))}{(\lambda_g - \lambda_b)(h\lambda_b(hr + f) - cf)}.$$

If a three phase equilibrium exists, it must be that play eventually transitions to an immediate-exit phase. Recall that in this phase $q^u \equiv 1$. Using (4) and (9), we obtain that, if uninformed players exert effort (9), a necessary condition for an informed player wanting to exit at a time t with $q^u(t) \equiv 1$ is

$$f \geq \lambda_b h \;\; \textit{or} \;\; q^g(t) \leq \overline{p}^I := \frac{(c - h\lambda_b)(\lambda_b(f + hr) + f\beta)}{(\lambda_g - \lambda_b)(h\lambda_b(hr + f) - cf) + f\beta(c - h\lambda_b)}. \tag{11}$$

We say that the project exhibits a *weak free-riding problem* at time t if (11) is satisfied. Otherwise, a project exhibits a *strong free-riding problem*.

Notice that if $F^{-1}(1) < 1$, then $F^{-1}(1) < \overline{p}^I$, and hence if the prior belief satisfies $p^g(0) \in \left[F^{-1}(1), \overline{p}^I\right]$ one may suspect that a three-phase equilibrium as well as an equilibrium

[20]In subsection B.3, we have identified necessary conditions for each of the phases to exist.

with only one, immediate-exit, phase may exist. However, as we show in Proposition B.1 that given the evolution of beliefs in each of the phases, for a prior in $\left[F^{-}1(1), \overline{p}^I\right]$ no three-phase equilibria can exist.

This implies that for all three phases to exist in equilibrium, it must be that $p^g(0) > \overline{p}^I$ and that the belief that the state is good must eventually decrease to $p^g(t) \leq \overline{p}^I$. Hence, it must be that $\overline{p}^I \in (0,1)$, which imposes a lower bound on the discount rate r:

$$r > \frac{\lambda_g f(c - h\lambda_b)}{h\lambda_b(h\lambda_g - c)}. \tag{12}$$

We say that players are *moderately patient.*

6.1 Three-Phase Equilibria

We now discuss projects that exhibits a strong free-riding problem, and show that the equilibrium with the longest duration of experimentation is a *three-phase equilbrium*: Play starts with a no-exit phase, then transitions through the gradual-exit to the immediate-exit phase. We discuss projects with a weak free-riding problem in subsection 6.2. In subsection 7.2, another class of equilibria are presented: symmetric equilibria that display earlier final-exit times, which can be interpreted as endogenous deadlines.

Proposition 6.1 (Three-Phase Equilibrium with the Longest Duration of Effort)**.** *Suppose that efforts are productive, the project exhibits a strong free-riding problem at* $t = 0$*, and players are moderately patient. Then there exist transition times* t^N*,* t^G *and a final exit time* $\overline{t}^I < \infty$*, such that there exists a symmetric perfect Bayesian equilibrium which consists of three phases: a no-exit phase,* $t \in \left[0,\, t^N\right)$*; a gradual-exit phase,* $t \in \left[t^N,\, t^G\right)$*; and an immediate-exit phase,* $t \in \left[t^G,\, \overline{t}^I\right)$*. Effort levels of uninformed players on the equillibrium path are given by* (7) *and* (9)*, respectively.*

Let us provide some intuition for this equilibrium.

An informed player knows that the state is bad. Initially his belief that his opponent is still uninformed and hence is exerting effort may be high enough such that the expected payoff from staying with the project is higher than the payoff from the outside option. If so, play starts with a no-exit phase, an informed player stays with the project and free-rides on the expected effort from his opponent. This is the case for projects that exhibit a strong free-riding problem.

As time passes, players become more pessimistic about their opponent still being uninformed. An uninformed player becomes more worried that the state is bad and his opponent is informed and free-riding. For an informed player, it becomes more likely that the other player is also informed, not exerting effort, and hence the project has reached a deadlock. The expected effort exerted by the opponent if the state is bad decreases over time and it becomes less attractive for an informed player to remain with the project.[21] At some threshold time $t^N \in [0, \infty)$, equilibrium play enters the gradual-exit phase. In the gradual-exit phase, an informed player is indifferent between staying and exiting, and exits at a finite rate. Observing that the opponent has not exited is good news and encourages uninformed players to keep exerting effort. However, when $\beta < \lambda_g - \lambda_b$, even if an uninformed player were certain that his opponent is also uninformed, he would become more pessimistic about the state being good as more effort is put into the project. To counter-balance this effect the equilibrium exit rate increases during the gradual-exit phase, eventually to infinity.[22] Consequently, if both players are still involved in the project at the end of the gradual-exit phase, then each player believes that his opponent is uninformed with probability one. At the same time, these uninformed players have become rather pessimistic about the state, and are not willing to exert high effort. The game proceeds to the immediate-exit phase: any player who becomes informed exits immediately, because the equilibrium effort is so low that the flow payoff from staying is strictly less than the level of the outside option, even if the opponent is uninformed and exerting effort with probability one. The immediate-exit phase lasts until the final exit time t^I at which all players opt for the outside option. At time t^I, players have become so pessimistic about the state being good that any player would choose the outside option if his opponent had opted out. The latest feasible final exit time $\bar{t}^I$ is the time at which the flow payoff of an uninformed player from staying drops to the level of the outside option. At this point any player would opt for the outside option, regardless of what his opponent does. This equilibrium with final exit time $\bar{t}^I$, is the one in which effort is sustained the longest. We refer to this equilibrium as the *equilibrium with the maximal duration of experimentation.*

As we will show the final exit time is not uniquely determined but there is an interval of feasible final exit times, each characterizing a symmetric equilibrium. Equilibria with such earlier final exit times – which act as an endogenous deadline – are discussed in subsection 7.2.

6.2 Immediate-Exit Equilibria

If projects exhibit a weak free-riding problem (11), in equilibrium, informed players exit immediately. There is no delay in information transmission. These *immediate-exit equilibria*

[21] This result is formally established in Lemma B.4.

[22] In the stationary case, there exists a two-phase equilibrium with a no-exit and a gradual-exit phase in which players' beliefs, the effort level and exit rate are constant in the gradual-exit phase. There is no immediate-exit phase or final exit time. The game only ends if a success or signal arrives after which all players exit immediately. A detailed discussion can be found in the online appendix.

consist of only one, the immediate-exit phase: If a player becomes informed, he immediately exits and takes the outside option. The situation is as if signals were public. Equilibrium effort levels are given by (9).

Proposition 6.2 (Immediate Exit Equilibrium with the Longest Duration of Effort)**.** *Consider a project that exhibits a weak free-riding problem and suppose that efforts are productive. Then there exists a final-exit time $\bar{t}^I$ and an immediate-exit equilibrium, such that, on the equilibrium path, informed players exert no effort and exit immediately. An uninformed player exerts effort a^I given by* (9)*.*

In an immediate-exit equilibrium, depending on the parameter region, effort levels may be interior throughout. For other parameter regions, players may initially exert full effort, either throughout until the final-exit time, or until their belief that the state is good becomes sufficiently low such that efforts (9) become interior. Details for this including formal conditions for each of these cases can be found in the appendix.

As shown in Corollary 5.1, the beliefs at which uninformed players exit depends on whether players are exerting full or interior effort at the final-exit time. If uninformed players exert full effort until the exit time, then $p^g(\bar{t}^I) = p_2^*$ – the cooperative threshold.[23]

As we show in the following corollary, for $f \geqslant c$ (recall that $c > \lambda_b h$), the coorperative solution is achieved as an equilibrium outcome: there is no delay in information transmission or effort.

Corollary 6.1. *If a project exhibits a weak free-riding problem, efforts are productive, and $f \geqslant c$, then the immediate-exit equilibrium with the longest duration of experimentation is outcome equivalent to the cooperative solution.*

7 Equilibrium Effort Levels and Endogenous Deadline

In this section, we discuss how equilibrium effort levels evolve over time with a focus on a new behavior that arises in our setting. Moreover, we examine (endogenous) deadlines – another new aspect in our setting: We will show that endogenous deadlines in the form of earlier final-exit times can exist and explain how they effect equilibrium effort levels.

7.1 Motion of Equilibrium Effort Levels

A natural question to ask is how the equilibrium effort levels evolve. Over time, if no success arrives, players become more pessimistic about the state being good. One may expect that

[23] If the prior belief $p^g(0)$ is lower than the cooperative threshold p_2^*, both players exit at time 0 in the immediate-exit equilibrium characterized by Proposition 6.2.

this results in decreasing equilibrium effort levels over time.[24] This intuition is correct for the gradual-exit and immediate-exit phase.

Lemma 7.1. *In a gradual- and immediate-exit phase, on the equilibrium path, effort levels are decreasing over time.*

It may come as a surprise that we find that in an initial no-exit phase, an uninformed player's effort level may increase over time.

Proposition 7.1 (Increasing effort in no-exit phase)**.** *The equilibrium effort level in the no-exit phase increases in t, if the markup of effort in the bad state is positive, $\lambda_b\left(h+\frac{f}{r}\right)>c$, and the arrival rate of a bad-state revealing signal is sufficiently high, that is, if*

$$\beta \geqslant \frac{2cr\left(\lambda_g-\lambda_b\right)^2}{\lambda_g\left(\lambda_b(f+hr)+cr\right)-2cr\lambda_b}. \tag{13}$$

To get some intuition for this result, we decompose the effect and consider how the effort level changes as a function of the beliefs q^g, q^u. Recall that both beliefs are decreasing in the no-exit phase.

First, observe that a lower belief q^g makes players less optimistic about the state, and less willing to put effort.[25] Therefore, the effect of increasing efforts must be driven by the relation between q^u and an uninformed player's effort level. To understand this relation, let us first consider the stationary case, with $\beta=\lambda_g-\lambda_b$. In this case, on path q^g remains constant. Therefore, conditional on the event that both players are still uninformed, player i remains indifferent among all effort levels, if his opponent's effort level remains the same as the effort at time 0. Conditional on the event that his opponent is informed and hence has stopped working, the uninformed player strictly prefers to exert effort if the markup of effort in the bad state is positive.[26]

The combined effect of these two events would make player i strictly prefer to exert effort, if the effort level of his uninformed opponent would remain the same as at time 0. Recall that players' effort inputs are substitutes. Therefore, to make player i indifferent among all effort levels, his uninformed opponent's effort level must increase over time.

Similarly, if the markup of effort in the bad state is negative, uninformed player i strictly prefers to shirk and exert no effort conditional on the event that his opponent is informed

[24] Previous literature such as Bonatti and Hörner (2011) finds decreasing efforts if players become more pessimistic over time.

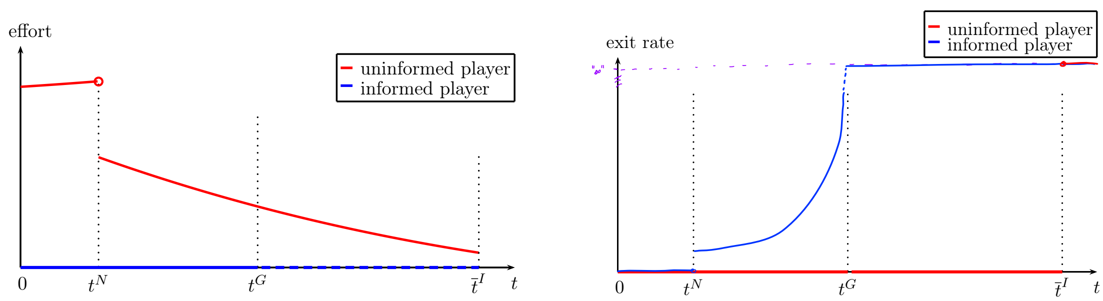
[25] Recall that the belief q^g is the probability that the state is good, conditional on both players being uninformed.

[26] Recall the discussion in section 3: in the no-exit phase, on path players do not exit. Hence, for the effort decision of the uninformed player, the situation is as if he is locked into the project.

and hence has stopped working. To counteract this incentive to shirk, the effort level of player i's uninformed opponent must decrease over time.

Let us now turn to the general case, $\beta < \lambda_g - \lambda_b$. If the markup of effort in the bad state is negative, then the effort in the no-exit phase must decrease in time. However, if the markup of effort in the bad state is positive, then it is unclear which force, the one through q^g or the one through q^u, dominates. When β is high enough (13), the belief q^g does not drop very fast. Therefore, the motion of the effort is largely affected by the force through q^u. Consequently, as shown in Proposition 7.1, the effort in the no-exit phase increases over time.[27]

Figure 4 illustrates an example of equilibrium effort levels and exit rate as a function of time. In this case, β is sufficiently large so that the effort level increases in the no-exit phase.



(a) Equilibrium effort levels.

(b) Equilibrium exit rates.

Figure 4: Three phase equilibrium.

To wrap up the discussion about effort levels, notice that at the transition time to the gradual-exit phase, t^N, there is a discontinuity – effort levels drop. Intuitively, when the game transitions from the no-exit to the gradual-exit phase, an uninformed player has more incentive to delay effort, since he can learn from observing whether or not his opponent exits. To counterbalance this effect, the effort level must drop at the transition time. The drop decreases the incentive of an uninformed player to delay effort, since his opponent's lower effort level reduces the benefit from postponing his own effort.

[27]To see that increasing efforts are indeed possible, note that in the limiting case $\beta = \lambda_g - \lambda_b$, as long as the markup of effort in the bad state is positive, condition (13) is satisfied. Therefore, if the markup of effort in the bad state is positive, the parameter region (of $\beta < \lambda_g - \lambda_b$) under which the effort in the no-exit phase increases has nonempty interior.

7.2 Endogenous Deadlines and Final-Exit Times

Until now, we focused on equilibria with the longest duration of experimentation. However, this is not the unique symmetric equilibrium in our model. In our setting, equilibria with earlier exit-times exist. One can interpret these exit times as endogenous deadlines, agreed upon by the players involved in the project. We now discuss these equilibria and the additional features that earlier exit-times create.

To gain some intuition for the existence of endogenous deadlines, recall that in section 6, we have identified symmetric equilibria with the longest duration of experimentation. In other words, the latest final exit time $\bar{t}^I$ after which no positive effort can be sustained and both players must exit. Notice moreover, that the belief at this time $\bar{t}^I$, is strictly lower the single-player threshold, at which an individual player would stop experimenting.[28] For the belief at this time it is not worthwhile for an individual player to stay and work on the project by himself. Hence, no player wants to deviate from the equilibrium strategy that prescribes for all players to exit at time $\bar{t}^I$. This consideration suggests that there may exist symmetric equilibria with earlier exit times. Our next result confirms this. For clarity of exposition we restrict attention to a parameter range in which only immediate-exit equilibria exist, and for equilibria with the longest duration of experimentation (section 6) effort levels would be interior throughout.[29] We show that under certain conditions, there exist an interval of (final-exit) beliefs and corresponding final-exit times $\hat{t} \in [\underline{t}^I,\ \bar{t}^I]$, such that for each of these final-exit times there exists a unique symmetric equilibrium.

Proposition 7.2 (Immediate Exit Equilibrium with Endogenous Deadlines)**.** *Consider a project that exhibits a weak free-riding problem, suppose that efforts are productive, and that* $p^g(0) < \overline{p}^I$ *and* $\bar{f} \leq f < \lambda_b h$*. Then, for any belief* $\widehat{p} \in \left(\frac{c-\lambda_b h}{h(\lambda_g - \lambda_b)},\ \min\{p_1^*,\ p^g(0)\}\right]$*, there exist a unique final-exit time* $t^I > 0$ *and a jump-time* $\widehat{t} \in \left[0, t^I\right)$ *such that there exists a symmetric immediate-exit equilibrium in which the effort level is given by*

$$a_i(t) = \begin{cases} a_i^I(t) & \text{for } t \in \left[0,\, \widehat{t}\right), \text{ and} \\ 1 & \text{for } t \in \left[\widehat{t}, t^I\right], \end{cases}$$

and the equilibrium belief at the final-exit time is $p^g(t^I) = \widehat{p}$*.*

Let us provide some intuition for this result. If the equilibrium strategies prescribe for players to exit at time t^I and the belief at the final-exit time t^I is below p_1^*, then no player

[28]This follows directly from Proposition 6.1 and Proposition 3.1 and since $\max\{\frac{c-h\lambda_b}{h(\lambda_g-\lambda_b)},\ p_2^*\} < p_1^*$.

[29]As established in **??**, $p^g(0) < \overline{p}^I$ and $\bar{f} \leq f < \lambda_b h$ guarantees that effort levels in the immediate-exit equilibirum with the longest duration of effort are interior throughout. Notice that this result can be generalized easily, since generically any three-phase equilibrium ends with an immediate-exit phase.

would want to deviate and stay at time t^I instead of exiting. Staying would mean for the player to now be working on the project as an individual, but for a belief below p_1^* it would be optimal for the player to exit. This explains why in our setting, there exist equilibria with earlier finite-exit times than $\bar{t}^I$. They can be interpreted as endogenous deadlines: If players agreed on exiting at time $t^I < \bar{t}^I$, as long as the belief at t^I falls into the range $\left(\frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)},\ \min\{p_1^*,\ p^g(0)\}\right]$, then none of the players wants to deviate.

These equilibria with endogenous deadlines, $t^I < \bar{t}^I$, exhibit a deadline effect similar to the one induced by exogenous deadlines in Bonatti and Hörner (2011). As a consequence of the earlier finite-exit time, there needs to be a jump-time $\hat{t} < t^I$ at which efforts jump to one. To see this, notice that if efforts would remain the same as in the equilibrium with the longest duration of experimentation, then at (right before) time $t^I < \bar{t}^I$, the flow payoff from staying with the project would be strictly higher than that of the outside option. Hence, instantenously before the exit-time, a player would want to deviate and exert higher effort. To counteract this, players must exert full effort right before the final-exit time t^I. Hence, there exists an interval $[\hat{t},\ t^I]$ during which players exert full effort, in anticipation of the approaching deadline. Earlier in the game, at times $t < \hat{t}$ the deadline is far enough in the future such that it has no effect on equilibrium effort levels – they are $a_i^I(t)$ as given by (9).

Notice that if $p^g(0) \in \left(\frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)},\ p_1^*\right]$, then a possible equilibrium is the one in which players do not take up the project at all, but exit at $t = 0$. However, as long as $p^g(0) > \frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}$, then there exist non-trivial equilibria, in which players exert effort over some non-empty time-interval.

8 Discussions

8.1 Off-path Beliefs and Behavior

Here, we briefly discuss players' behavior off path. Suppose that an uninformed player deviated in such a way that, at time t, the aggregate effort of player i over the interval $[0, t)$ is lower than it would have been on path. This means that player i is more optimistic than he would have been on path. His optimism leads him to exert maximal effort until the time at which his private belief reverts to the common belief. At this time he reverts to the common strategy. If a player deviates in such a way that his realized aggregate effort is greater than in equilibrium, he is more pessimistic and provides no effort until the private belief reverts to the common belief again. Regardless of his past deviation, an informed player assigns the same belief to the event that his opponent is informed. Therefore, off path, it is still optimal for him to follow the equilibrium exiting strategy.

If the opponent has not exited by time $t > t^I$, i.e., after the final-exit time, then an informed player believes that his opponent is exerting zero effort, and thus the informed player exits immediately. An uninformed player also believes that his opponent is exerting zero effort and decides whether to exit based on his private belief that the state is good. This private belief is calculated based on his own and his opponent's aggregate effort over the interval $\left[0, t^I\right)$. For time t^I to be an equilibrium final exit time it must be that an (uninformed) player's belief at this time is below the single-player threshold. Otherwise uninformed players would want to deviate and continue pursuing the project by themselves instead of exiting. Hence, off-path it is also optimal for an uninformed player to exit immediately at any time $t > t^I$ at which his opponent has not exited yet.

8.2 Comparative statics

Consider a given set of parameters λ_g, λ_b, β, c, h, r and a fixed prior probability $p^g(0)$ such that efforts are individually productive for all $f \geq 0$, that is, $p^g(0) \geqslant \frac{c-h\lambda_b}{h(\lambda_g-\lambda_b)}$. Figure 5 illustrates which type of symmetric equilibria with maximal duration of experimentation exist, as f increases. Given a fixed prior, the blue dashed line illustrates equilibrium properties as f increases. For low values of the outside option, equilibria have a three-phase structure as identified in Proposition 6.1. The payoff of the outside option is low enough such that an informed player chooses to stay with the project and delays his exit decision. This leads to delayed information transmission. Recall that the role of the outside option f on informed players' exit-decisions and uninformed players' effort is quite different (subsection 3.1). For low f, uninformed players have an incentive to delay effort and hence only exert interior effort. Hence, in this parameter region there are two types of inefficiencies: *delayed information transmission* and *delayed effort*. As f increases, the outside option becomes more attractive and the project turns into one that exhibits a weak free riding problem (11). For payoffs $f \geq f_1$, where f_1 is characterized by (11), informed players exit immediately. The inefficiency due to delayed information transmission disappears. Depending on the prior, uninformed players may still only exert interior effort, or initially exert full effort but efforts will be interior at the final-exit time (for $f \geq f_2$). The inefficiency due to delayed effort remains in the team problem. A further increase in f makes the outside option more attractive, and hence diminishes an uninformed player's incentive to delay effort. As f increases, for payoffs $f \geq c$ uninformed players exert full effort throughout. At this point there are no inefficiencies in the team problem anymore. Both inefficiencies, delayed information transmission and delayed effort have disappeared. Finally, if the payoff of the outside option is so high that the prior probability is below the cooperative threshold, then both players exit immediately.

This is the case if $f \geq f_3$ where f_3 is the value at which $p^g(0) = p_2^*(f)$. Both f_1 and f_3

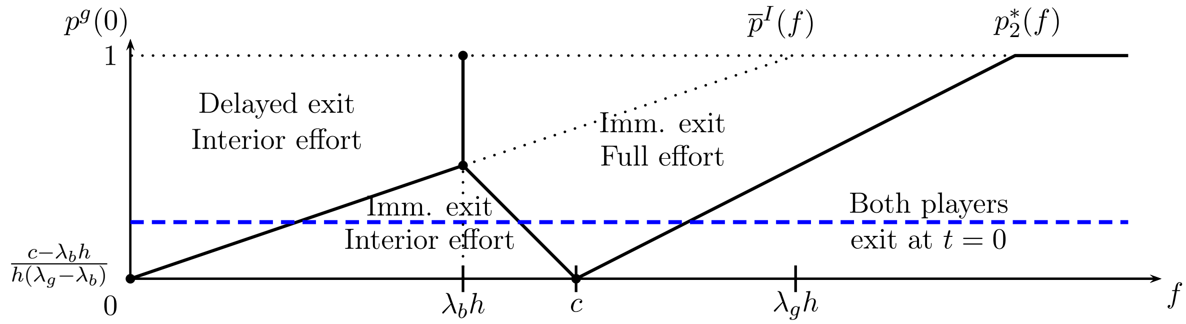


Figure 5: Classification of symmetric equilibria with maximal duration of experimentation.

are increasing functions of $p^g(0)$, while f_2 decreases in $p^g(0)$. Moreover, as can be seen from Figure 5, depending on the given prior $p^g(0)$, as f increases, not all types of equilibria need to exist.

8.3 Communication

Our model can be adjusted to accommodate communication. Not allowing for verifiable disclosure or cheap talk is without loss in our case, since an informed player will never reveal that he has received bad news.

To see this, first consider the option of verifiable disclosure of information. Assume that an informed player can publicly disclose, at any time, the private signal that he has obtained. Once an informed player publicly discloses a signal, no player will exert any more effort, and hence all players will exit immediately. The outcome is the same as if this informed player had chosen to exit. Hence, allowing for verifiable disclosure has no impact on the equilibrium that we have characterized.

To incorporate cheap-talk communication, assume that each player can send a public message, declaring whether he is informed or not, at any time in the game. Again, an informed player doesn't want to demotivate his opponent, and will announce that he is uninformed unless he is ready to exit. An uninformed player also has no incentive to say that he is informed. Hence, allowing for cheap-talk communication also has no impact on the equilibrium that we have characterized.

8.4 More players

Our analysis can be adjusted to allowing for n players with $n > 2$. Similar arguments as in section 4 will characterize the equilibrium effort level and exit rate. Take the no-exit phase as an example. We again consider the effect of shifting ε effort from today to tomorrow, from an uninformed player i's perspective. This ε effort will not be carried out, if a success or a signal arrives today. In order to calculate the probability that a success or a signal arrives today, an uninformed player i must keep track of the probability that k out of his $n-1$ opponents are uninformed for each $k \in \{0, 1, ..., n-1\}$. This is the main difference to the two-player case, in which player i only keeps track of the probability that player j is uninformed. Hence, an uninformed player's equilibrium effort can be derived from inequalities similar to (7) and (10).

Due to the high dimensionality of the belief space, it is hard to establish comparative statics result with respect to n. Yet, simulation shows that similar comparative statics result as in the literature arise in our setting as well. Players benefit from a larger team due to the positive externality from each others efforts, even though more players tend to make the free-riding problem more severe.

Appendix

A Preliminary results

Lemma A.1. *In any equilibrium, an informed player exerts zero effort.*

Proof. Suppose that player i is informed at time t. If player j has already left at this time t, then player i shall exit as well since the net benefit of effort is negative in the bad state. If player j is still with the project at t, and player i decides to stay over $[t, t + \mathrm{d}t)$, we argue that player i exerts no effort. Suppose that player i exerts effort a_i over $[t, t + \mathrm{d}t)$. We let a_j be the expected effort by player j, and d_j the probability that player j exits over $[t, t + \mathrm{d}t)$. If a success occurs or player j exits in $[t,\ t + \mathrm{d}t)$, player i optimally takes the outside option at $t + \mathrm{d}t$. Conditional on reaching $t + \mathrm{d}t$ without a success or player j's exit, player i updates his belief that player j is uninformed. Let $q^u(t + \mathrm{d}t)$ be this belief and $V(t + \mathrm{d}t)$ be player i's continuation payoff. A key observation is that the belief $q^u(t + \mathrm{d}t)$ does not depend on player i's own effort a_i, and hence the continuation payoff $V(t + \mathrm{d}t)$ does not depend on a_i either. Player i's payoff at time t is:

$$r\left(\lambda_b h(a_i + a_j) - c a_i\right)\,\mathrm{d}t + e^{-r\,\mathrm{d}t}\left\{(\lambda_b(a_i + a_j)\,\mathrm{d}t + d_j)(f - V(t + \mathrm{d}t)) + V(t + \mathrm{d}t)\right\}.$$

Given that $\lambda_b h < c$ and $f \leqslant V(t + \mathrm{d}t)$, this payoff strictly decreases in a_i. So, player i puts no effort. □

Lemma A.2. *In any equilibrium, at any time t, if an informed player prefers to stay, then an uninformed player strictly prefers to stay.*

Proof. Suppose that an informed player weakly prefers to stay at time t. We argue that if an uninformed player uses the same continuation strategy as an informed player does, the uninformed player's payoff is strictly higher. Based on Lemma A.1, an informed player exerts no effort and decides when to exit. An informed player's payoff consists of the payoff generated by his opponent's effort and the payoff from the outside option after he exits. If an informed player weakly prefers to stay, he expects his opponent, if uninformed, to exert strictly positive effort. An uninformed player has a strictly higher belief that the state is good and a strictly higher belief that his opponent is uninformed. Therefore, if an uninformed player uses the same strategy as an informed one, his payoff is strictly higher. It follows immediately that if an informed player prefers to stay, then an uninformed player strictly prefers to stay. □

B Proofs

B.1 Proofs of Section 3

Proof of Proposition 3.1. In the cooperative game, it is without loss to focus on symmetric strategies. We let $\overline{V}$ denote the cooperative payoff per player.

If $n\lambda_b h - c \geq f$, then it is optimal to exert full effort until a success occurs. The payoff $\overline{V}$ equals:

$$\overline{V}(p^g(0)) = p^g(0)\frac{n\lambda_g(hr+f)-cr}{n\lambda_g+r} + (1-p^g(0))\frac{n\lambda_b(hr+f)-cr}{n\lambda_b+r}.$$

If $n\lambda_b h - c < f$, then it is optimal for an informed player to exit immediately. This is as if the signals were public. The belief of state g, in the absence of a success or a signal, evolves according to (1). Given the belief p^g of state g, the flow payoff per player if all players choose the effort level $\tilde{a}$ is $(n\lambda^s(p^g)h - c)\,\tilde{a}$. By the Principle of Optimality, the value function of the cooperative game satisfies

$$\overline{V}(p^g) = \max_{\tilde{a}\in[0,1]} \left\{ r\left(n\lambda^s(p^g)h - c\right)\tilde{a}\ \mathrm{d}t + e^{-r\mathrm{d}t}\left(n\lambda^{s,i}(p^g)\,\tilde{a}\ \mathrm{d}t\left(f - \overline{V}(p^g + \ \mathrm{d}p^g)\right)\right) + \overline{V}(p^g + \ \mathrm{d}p^g)\right\}.$$

Substituting $\overline{V}(p^g + \ \mathrm{d}p^g) = \overline{V}(p^g) - \dot{\overline{V}}(p^g)n(1-p^g)p^g(\lambda_g - \lambda_b - \beta)\tilde{a}\ \mathrm{d}t$, using $1 - r\ \mathrm{d}t$ as an approximation to $e^{-r\ \mathrm{d}t}$ and rearranging, we obtain the Bellman equation:

$$\overline{V}(p^g) = \max_{\tilde{a}\in[0,1]} \left\{ (n\lambda^s(p^g)h - c)\tilde{a} + \frac{n\lambda^{s,i}(p^g)\tilde{a}}{r}\left(f - \overline{V}(p^g)\right) - \frac{n(1-p^g)p^g(\lambda_g-\lambda_b-\beta)\tilde{a}}{r}\dot{\overline{V}}(p^g)\right\}.$$

The linearity in $\tilde{a}$ of the maximand in the Bellman equation immediately implies that it is always optimal to choose either $\tilde{a} = 0$ or $\tilde{a} = 1$. In the latter case, $\overline{V}$ satisfies the first-order ODE:

$$\overline{V}(p^g) = n\lambda^s(p^g)h - c + \frac{1}{r}\left\{ n\lambda^{s,I}(p^g)\left(f - \overline{V}(p^g)\right) - n(1-p^g)p^g(\lambda_g - \lambda_b - \beta)\dot{\overline{V}}(p^g)\right\}.$$

Let p_n^* denote the cutoff belief at which players are indifferent between staying with the project while exerting full effort and taking the outside option. The value matching $\overline{V}(p_n^*) = f$ and smooth pasting $\dot{\overline{V}}(p_n^*) = 0$ conditions allow us to solve for the cutoff belief p_n^* and the constant of the integration in the solution to the above ODE. The cooperative threshold p_n^* satisfies

$$nh\left(\lambda_g p_n^* + (1-p_n^*)\lambda_b\right) - c = f.$$

If the belief is above the cooperative threshold, players stay with the project and exert full effort. Otherwise they take the outside option. □

B.2 Proofs of Section 4

We set up the general problem first to then prove the cases of Lemma 4.1 (effort in the no-exit and general-exit phase) and Lemma 4.2 (effort in the immediate exit phase) separately. In both cases, we want to show that, if player j follows the suggested strategy, then player i finds it optimal to choose the suggested effort and exit levels.

First, consider a player's optimal action after a success or exit of the opponent. Given that there is only one success, it is optimal for any player to exit immediately after a success. Moreover, recall that the suggested strategy profile, only informed players exist. Hence, if player i observes his opponent exit then he updates his belief to $p^g(t) = 0$, that is, player i knows that the state is bad in which case it is optimal for him to exit immediately.

Now, consider the case in which no success or exit has occurred yet. We take the behavior of informed players as given, and consider each of the possible behaviors as separate cases: no-exit, gradual-exit (instantaneous exit with positive but finite probability), and immediate-exit (instantaneous exit with certainty). Hence, in what follows, we determine optimal behavior player i if he is *uninformed*. To verify that player i finds it optimal to follow the suggested strategy, we formulate player i's problem as a control problem with free endpoint:

Conditional on no success and no exit of the opponent, let $\tilde{p}^g(t)$, $p^{b,ii}(t)$, $p^{b,iu}(t)$, $p^{b,ui}(t)$, $p^{b,uu}(t)$ denote the following probabilities: (i) the state is good; (ii) the state is bad and both are informed; (iii) the state is bad and only i is informed; (iv) the state is bad and only j is informed; (v) the state is bad and no player is informed. If player j exerts effort $a_j(t)$, then player i's flow payoff (net of f) at time t is given by

$$\begin{aligned}\zeta(t) =& h\left[\left(\lambda_g \tilde{p}^g(t) + \lambda_b p^{b,uu}(t)\right)\left(\tilde{a}_i(t) + a_j(t)\right) + \lambda_b \tilde{a}_i(t) p^{b,ui}(t)\right] \\ &- \left(c\tilde{a}_i(t) + f\right)\left(\tilde{p}^g(t) + p^{b,ui}(t) + p^{b,uu}(t)\right) \\ &+ \left(p^{b,ii}(t) + p^{b,iu}(t)\right) \max\left\{0, \frac{p^{b,iu}(t)}{p^{b,ii}(t) + p^{b,iu}(t)} a_j(t)\lambda_b h - f\right\}.\end{aligned} \tag{14}$$

Here, the first entry of the last term is an informed player's payoff if he exits, the second entry is the informed player's payoff if he remains with the project, does not exert effort but free-rides on player j's expected effort.[30]

[30] Recall that, as discussed in Appendix A, it is never optimal for an informed player to exert effort.

We define two state variables

$$w_1(t) = e^{-\lambda_g \int_0^t \tilde{a}_i(s)\mathrm{d}s}, \quad w_2(t) = e^{-(\lambda_b+\beta)\int_0^t \tilde{a}_i(s)\mathrm{d}s},$$

and let $\gamma_1(t)$ and $\gamma_2(t)$ be the associated costate variables. For ease of exposition, we also let

$$x_1(t) = e^{-\lambda_g \int_0^t a_j(s)\mathrm{d}s}, \quad x_2(t) = e^{-(\lambda_b+\beta)\int_0^t a_j(s)\mathrm{d}s}.$$

Since a_j is given, $x_1(t)$ and $x_2(t)$ are given functions of time. Substituting $w_1'(t) = -\lambda_g \tilde{a}_i(t) w_1(t)$ and $w_2'(t) = -(\beta+\lambda_b)\tilde{a}_i(t)w_2(t)$, we obtain the Hamiltonian of this problem:

$$\mathcal{H}\left(\tilde{a}_i, w_1, w_2, \gamma_1, \gamma_2, t\right) = e^{-rt}\zeta(t) - \tilde{a}_i(t)\left[(\beta+\lambda_b)\gamma_2(t)w_2(t) + \lambda_g\gamma_1(t)w_1(t)\right]. \tag{15}$$

Proof of Lemma 4.1.

No-exit phase. During the no-exit phase, say $t \in [t_0,\ t_1)$, the probabilities $\tilde{p}^g(t), p^{b,ui}(t), p^{b,uu}(t)$ are given as follows:

$$\tilde{p}^g(t) = \tilde{p}^g(0)w_1(t)x_1(t), \quad p^{b,ui}(t) = (1-\tilde{p}^g(0))\frac{\beta w_2(t)(1-x_2(t))}{\beta+\lambda_b}, \text{ and} \tag{16}$$
$$p^{b,uu}(t) = (1-\tilde{p}^g(0))w_2(t)x_2(t).$$

Substituting these probabilities into $\mathcal{H}\left(\tilde{a}_i, w_1, w_2, \gamma_1, \gamma_2, t\right)$, we obtain that the Hamiltonian is linear in the state variables w_1, w_2.[31] The derivative $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)}$ equals zero for all $t \in [t_0, t_1)$ if and only if both $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ and $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)}$ equal zero for all $t \in [t_0, t_1)$. Substituting $\gamma_1', \gamma_2', w_1', w_2', x_1', x_2'$ into the equation $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t} = 0$, we obtain the equilibrium effort in the no-exit phase – if interior – is

$$a_j^N(t) = \frac{\tilde{p}^g(t)(\lambda_g(hr+f)-cr) + (p^{b,ui}(t)+p^{b,uu}(t))(\lambda_b(hr+f)-cr)}{c(\lambda_b p^{b,uu}(t) + \lambda_g \tilde{p}^g(t))}.$$

This corresponds to the formula (8) that we obtain from the heuristic argument in section 4.[32] Notice that the equation defining the effort level does not depend on β, but the motion of beliefs in (26) depends on β. The arrival rate of the private signal only affects a player's effort level indirectly through the motion of beliefs.

In the optimum the first-order condition, $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)} = 0$, must be satisfied, which requires that

[31] It will turn out that the Hamiltonian is linear in w_1, w_2 during the gradual-exit and immediate-exit phase as well.

[32] Notice that in the proof we consider probabilities conditional on no success or exit, where in the main part of the paper we condition in addition on the event that player i is uninformed. Hence, it holds that $p^g = \frac{\tilde{p}^g}{\tilde{p}^g + p^{b,uu} + p^{b,ui}}$ and $p^{bu} = \frac{p^{b,uu}+p^{b,ui}}{\tilde{p}^g + p^{b,uu}+p^{b,ui}}$.

for all $t \in [t_0, t_1)$

$$(\beta + \lambda_b)\gamma_2(t)w_2(t) + \lambda_g\gamma_1(t)w_1(t) = e^{-rt}((h\lambda_b - c)(p^{b,ui}(t) + p^{b,uu}(t)) + \tilde{p}^g(t)(h\lambda_g - c)). \quad (17)$$

It is straightforward to check that this equality is satisfied, given the state- and co-state variables, equilibrium effort $a^N(t)$, and the belief system (16).

Gradual-exit phase. During a gradual-exit phase, say $t \in [t_1,\ t_2)$, an informed player i is indifferent between exiting and not. It must hold that

$$\frac{p^{b,iu}(t)}{p^{b,ii}(t) + p^{b,iu}(t)} a_j(t)\lambda_b h = 0. \quad (18)$$

The evolution of $\tilde{p}^g(t), p^{b,uu}(t)$ is the same as in the no-exit phase. The evolution of $p^{b,ui}(t)$ incorporates player j's exit behavior:

$$p^{b,ui}(t) = w_2(t)e^{D_j(t)}\left(\beta(1 - \tilde{p}^g(0))\int_{t_1}^{t} e^{-D_j(s)}a_j(s)x_2(s)\mathrm{d}s + \frac{p^{b,ui}(t_1)}{w_2(t_1)}\right),$$

with $D_j(t) = -\int_{t_1}^{t} d_j(s)\mathrm{d}s$. Substituting these probabilities and (18) into $\mathcal{H}(\tilde{a}_i, w_1, w_2, \gamma_1, \gamma_2, t)$, we obtain that the Hamiltonian is linear in the state variables w_1, w_2. The derivative $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)}$ equals zero for all $t \in [t_1, t_2)$ if and only if both $\frac{\partial(\partial \mathcal{H}/\partial \tilde{a}_i(t))}{\partial t}$ and $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)}$ equal zero for all $t \in [t_1, t_2)$. Substituting $\gamma_1', \gamma_2', w_1', w_2', x_1', x_2'$ into the equation $\frac{\partial(\partial \mathcal{H}/\partial \tilde{a}_i(t))}{\partial t} = 0$, we obtain the equilibrium exit rate:

$$\begin{aligned} d_j^G(t) =& \frac{p^{b,uu}(t)(f(\beta + \lambda_b) + h\lambda_b r - \lambda_b a_j(t)(\beta h + c) - cr) + \tilde{p}^g(t)(\lambda_g(hr + f) - c(\lambda_g a_j(t) + r))}{p^{b,ui}(t)(c - h\lambda_b)} \\ &+ \frac{(f(\beta + \lambda_b) - cr + h\lambda_b r)}{(c - h\lambda_b)}. \end{aligned}$$

This corresponds to the formula (8) that we obtain from the heuristic argument, if we solve for d_j. Combining this with (18), we also obtain the same formula for the effort level $a^G(t)$ that we obtain from the heuristic argument. In the optimum the first-order condition, $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)} = 0$, must be satisfied. This requires that for all $t \in [t_1,\ t_2)$

$$(\beta + \lambda_b)\gamma_2(t)w_2(t) + \lambda_g\gamma_1(t)w_1(t) = e^{-rt}((h\lambda_b - c)(p^{b,ui}(t) + p^{b,uu}(t)) + \tilde{p}^g(t)(h\lambda_g - c)), \quad (19)$$

which again is straightforward to verify. □

Proof of Lemma 4.2.

Immediate-exit phase. During an immediate-exit phase, say $t \in [t_2,\ t_3)$, if a player observes no exit of his opponent, he believes that his opponent is uninformed. Thus, $p^{b,ui}(t)$ remains zero for all $tt \in [t_2,\ t_3)$. Player i's flow payoff (net of f) at time t is given by

$$h\left[\left(\lambda_g \tilde{p}^g(t) + \lambda_b p^{b,uu}(t)\right)\left(\tilde{a}_i(t) + a_j(t)\right)\right] - \left(c\tilde{a}_i(t) + f\right)\left(\tilde{p}^g(t) + p^{b,uu}(t)\right).$$

The derivative $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)}$ equals zero for all $t \in [t_2,\ t_3)$ if and only if both $\frac{\partial(\partial \mathcal{H}/\partial \tilde{a}_i(t))}{\partial t}$ and $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)}$ equal zero for all $t \in [t_2,\ t_3)$. Substituting $\gamma_1', \gamma_2', w_1', w_2', x_1', x_2'$ into the equation $\frac{\partial(\partial \mathcal{H}/\partial \tilde{a}_i(t))}{\partial t} = 0$, we obtain the equilibrium effort level:

$$a_j^I(t) = \frac{p^{b,uu}(t)(f(\beta+\lambda_b) + r(h\lambda_b - c)) + \tilde{p}^g(t)(\lambda_g(hr+f) - cr)}{c(\beta+\lambda_b)p^{b,uu}(t) + c\lambda_g \tilde{p}^g(t)}.$$

This corresponds to the formula (9) that we obtain from the heuristic argument. The condition $\frac{\partial \mathcal{H}}{\partial \tilde{a}_i(t)} = 0$ requires that for $t \in [t_2,\ t_3)$

$$(\beta+\lambda_b)\gamma_2(t)w_2(t) + \lambda_g \gamma_1(t) w_1(t_2) = e^{-rt}(p^{b,uu}(t)(h\lambda_b - c) + \tilde{p}^g(t)(h\lambda_g - c)).$$

This condition is consistent with (17) and (19) since $p^{b,ui}(t)$ equals zero for $t \in [t_2,\ t_3)$, and hence it is straightforward to verify. □

B.3 Necessary Conditions for the Equilibrium Phases to Exist

The next lemma characterizes necessary conditions for a no-exit equilibrium phase or a gradual-exit equilibrium phase to exist. To simplify presentation, we define the function:

$$F(q^g) := \frac{(1-q^g)\lambda_b(f+hr)\left(c - h\lambda_b\right) + cfq^g\lambda_g}{hq^g\lambda_b\left(\lambda_g(f+hr) - cr\right)}. \tag{20}$$

The function $F(q^g)$ decreases in q^g as long as $cr < \lambda_g(f+hr)$, which is implied by $c < \lambda_g h$. Since F is decreasing, it's invertible with

$$F^{-1}(q^u) = \frac{\lambda_b(\lambda_g h - c)(f+hr)}{(\lambda_b h(f+hr) - cf)(q^u\lambda_g - \lambda_b) + (1-q^u)(h\lambda_b cr - cf\lambda_g)} \tag{21}$$

Moreover, define the function

$$\bar{f}(q^g) := \frac{rh\lambda_b(h\lambda^s(q^g) - c)}{\lambda^{s,i}(q^g)(c - h\lambda_b)}, \tag{22}$$

which is increasing in q^g.

Lemma B.1 (Necessary conditions for a no-exit/gradual-exit phase to exist)**.**

(i) No-exit phase: *There exists an a_j satisfying* (6) *with $d_j = 0$, and $\lambda_b h \cdot q^u a_j \geqslant f$, iff* $q^u \geqslant \max\left\{\frac{f}{\lambda_b h}, F(q^g)\right\}$.

(ii) Gradual-exit phase: *There exist an $a_j = \frac{f}{\lambda_b h q^u} \in [0,1]$ and $d_j \geqslant 0$ satisfying* (6) *iff* $q^u \geqslant \max\left\{\frac{f}{\lambda_b h}, F(q^g)\right\}$.

(iii) Finite-exit phases: *There exists $(q^u, q^g) \in [0,1]\times[0,1)$ such that $q^u \geqslant \max\left\{\frac{f}{\lambda_b h}, F(q^g)\right\}$ iff:*

$$f < \min\left\{\lambda_b h,\ \bar{f}(1)\right\}. \tag{23}$$

Under this condition, both $F(1)$ and $F^{-1}(1)$ are in $(0,1]$.

(iv) Under condition (23), *$\frac{f}{\lambda_b h} \leqslant F(q^g)$ holds for all q^g, iff*

$$r \leqslant \frac{(c-f)\lambda_g}{h\lambda_g - c}. \tag{24}$$

Proof. We first prove part (i). Substitute a_j given by (6) with $d_j = 0$, into $\lambda_b h \cdot q^u a_j \geqslant f$. This condition is equivalent to $\lambda_b h \cdot q^u \geqslant f$ if $a_j = 1$ and for $a_j < 1$ obtain

$$\lambda_b h \cdot q^u \frac{\lambda^s(p^g)(hr+f) - cr}{\lambda^u(p^g, p^{bu})c} \geqslant f,$$

which is equivalent to $q^u \geqslant F(q^g)$.[33]

We next prove the "if" part of (ii). If $q^u \geqslant \max\left\{\frac{f}{\lambda_b h}, F(q^g)\right\}$, then

$$\lambda_b h \cdot q^u \frac{\lambda^s(p^g)(hr+f) - cr}{\lambda^u(p^g, p^{bu})c} \geqslant f.$$

We can choose some $d_j \geqslant 0$ such that:

$$\lambda_b h \cdot q^u \frac{\lambda^s(p^g)(hr+f) - cr - p^{bi}(c - h\lambda_b)d_j}{\lambda^u(p^g, p^{bu})c} = f.$$

Moreover, since $q^u \geqslant \frac{f}{\lambda_b h}$, the resulting $a_j = \frac{f}{\lambda_b h \cdot q^u} \leqslant 1$.

[33] We obtain $q^u \geq \max\{\frac{f}{\lambda_b h},\ F(q^g)\}$, since if $\frac{\lambda^s(p^g)(hr+f)-cr}{\lambda^u(p^g,p^{bu})c} \geqslant 1$, then $a_j = 1$ and $q^u \geq \frac{f}{\lambda_b h} \geqslant F(q^g)$. If $\frac{\lambda^s(p^g)(hr+f)-cr}{\lambda^u(p^g,p^{bu})c} \leq 1$, then $q^u \overset{!}{\geqslant} F(q^g) \geqslant \frac{f}{\lambda_b h}$.

We next prove the "only if" part of (ii). We want to show that if $q^u < \max\left\{\frac{f}{\lambda_b h}, F(q^g)\right\}$, then we cannot find $a_j = \frac{f}{\lambda_b h \cdot q^u} \in [0,1]$ and $d_j \geqslant 0$ satisfying (6). Indeed, if $q^u < \frac{f}{\lambda_b h}$, then $a_j = \frac{f}{\lambda_b h \cdot q^u} > 1$. If $q^u < F(q^g)$, then

$$\lambda_b h \cdot q^u \frac{\lambda^s(p^g)(hr+f) - cr}{\lambda^u(p^g, p^{bu})c} < f,$$

which implies that for any $d_j \geqslant 0$ it holds that

$$\lambda_b h \cdot q^u \frac{\lambda^s(p^g)(hr+f) - cr - p^{bi}(c - h\lambda_b)d_j}{\lambda^u(p^g, p^{bu})c} < f.$$

We next prove part (iii). The condition (23) in part (iii) follows from the fact that $F(q^g)$ decreases in q^g so $q^u \geqslant F(q^g)$ is satisfied for some $(q^u, q^g) \in [0,1]^2$ iff $q^u \geqslant F(q^g)$ holds at $q^u = q^g = 1$. Given (23), both

$$F(1) = \frac{cf\lambda_g}{h\lambda_b\left(\lambda_g(f+hr) - cr\right)}, \quad \textit{and} \quad F^{-1}(1) = \frac{\lambda_b(f+hr)\left(c - h\lambda_b\right)}{\left(\lambda_g - \lambda_b\right)\left(h\lambda_b(f+hr) - cf\right)},$$

are smaller than one. We want to show that $F(1) > 0$ and $F^{-1}(1) > 0$. It is obvious that $F(1) > 0$. The inequality $F^{-1}(1) > 0$ is equivalent to $h\lambda_b(f+hr) > cf$, which follows from (23).

Lastly, to prove part (iv), notice that $\frac{f}{\lambda_b h} \leqslant F(q^g)$ holds for all q^g iff it holds for $q^g = 1$ because $F(q^g)$ strictly decreases in q^g. The condition $\frac{f}{\lambda_b h} \leqslant F(1)$ is equivalent to (24). □

The next lemma characterizes a necessary condition for an immediate-exit equilibrium phase to exist.

Lemma B.2 (Necessary condition for immediate-exit phase to exist)**.** *For an immediate-exit phase, $q^u = 1$ and $p^g = q^g$. There exists a_j satisfying (9) and $\lambda_b h \cdot a_j \leqslant f$ iff*

$$f \geqslant \min\{\lambda_b h,\ \bar{f}(q^g)]. \tag{25}$$

Notice that at $t = 0$, (25) is equivalent to a project exhibiting a weak free-riding problem.

Proof. This follows from substituting a_j given by (9) into $\lambda_b h \cdot a_j \leqslant f$. □

B.4 Proofs of Section 5

Proof of Lemma 5.1. First, notice that the evolution of beliefs in an immediate-exit phase is straightforward. Since informed players exit immediately, $q^u = 1$ throughout, and the evolution of p^g is given by (1): p^g is strictly decreasing if $\lambda_g - \lambda_b > \beta$; it is constant if $\lambda_g - \lambda_b = \beta$.

Next, we consider the belief evolution in the no-exit and gradual-exit phase. Here, we restrict attention to $q^u \geqslant \max\left\{\frac{f}{\lambda_b h}, F(q^g)\right\}$, since this is a necessary condition for these equilibrium phases to exist (cf Lemma B.1).
Consider the interval $[t, t + \mathrm{d}t)$. Let p^g, p^{bi}, p^{bu} be uninformed player i's beliefs at t. Suppose that this uninformed player i's effort is a_i over $[t, t + \mathrm{d}t)$. Suppose that player j's effort is a_j if he is uninformed, and his exit rate is d_j if he is informed.

Uninformed player i's updated beliefs at $t + \mathrm{d}t$ if he observes no success, signal, or player j's exit, are:

$$\begin{aligned}
p^g(t + \mathrm{d}t) &= \frac{p^g e^{-\lambda_g(a_i+a_j)\,\mathrm{d}t}}{p^{bi}e^{(-(\lambda_b+\beta)a_i-d_j)\,\mathrm{d}t} + p^{bu}e^{(-\lambda_b(a_i+a_j)-\beta a_i)\,\mathrm{d}t} + p^g e^{-\lambda_g(a_i+a_j)\,\mathrm{d}t}},\\
p^{bi}(t + \mathrm{d}t) &= \frac{p^{bi}e^{(-(\lambda_b+\beta)a_i-d_j)\,\mathrm{d}t} + p^{bu}\left(1 - e^{-\beta a_j\,\mathrm{d}t}\right)e^{(-\lambda_b(a_i+a_j)-\beta a_i)\,\mathrm{d}t}}{p^{bi}e^{(-(\lambda_b+\beta)a_i-d_j)\,\mathrm{d}t} + p^{bu}e^{(-\lambda_b(a_i+a_j)-\beta a_i)\,\mathrm{d}t} + p^g e^{-\lambda_g(a_i+a_j)\,\mathrm{d}t}},\\
p^{bu}(t + \mathrm{d}t) &= \frac{p^{bu}e^{(a_i+a_j)(-\lambda_b-\beta)\,\mathrm{d}t}}{p^{bi}e^{(-(\lambda_b+\beta)a_i-d_j)\,\mathrm{d}t} + p^{bu}e^{(-\lambda_b(a_i+a_j)-\beta a_i)\,\mathrm{d}t} + p^g e^{-\lambda_g(a_i+a_j)\,\mathrm{d}t}}.
\end{aligned} \tag{26}$$

This implies:

$$\begin{aligned}
q^u(t + \mathrm{d}t) &= \frac{q^u e^{-\beta a_j\,\mathrm{d}t}}{q^u + (1 - q^u)e^{\lambda_b a_j - d_j\,\mathrm{d}t}},\\
q^g(t + \mathrm{d}t) &= \frac{q^g e^{-(\lambda_g-\lambda_b-\beta)(a_i+a_j)\,\mathrm{d}t}}{1 - q^g\left(1 - e^{-(\lambda_g-\lambda_b-\beta)(a_i+a_j)\,\mathrm{d}t}\right)}.
\end{aligned}$$

The derivatives of q^u and q^g are:

$$\begin{aligned}
\dot{q}^u &= q^u(1 - q^u)d_j - q^u\left(\beta + (1 - q^u)\lambda_b\right)a_j,\\
\dot{q}^g &= -(1 - q^g)q^g(a_i + a_j)\left(\lambda_g - \lambda_b - \beta\right).
\end{aligned} \tag{27}$$

It follows that in the no-exit and gradual-exit phase, q^g is strictly decreasing if $\lambda_g - \lambda_b > \beta$ and is constant if $\lambda_g - \lambda_b = \beta$. Moreover, in the no-exit phase, d_j equals zero, so q^u strictly decreases.

In the gradual-exit phase, the sign of $\dot{q}^u$ depends on the equilibrium effort level and exit rate. For any $q^u > \frac{f}{\lambda_b h}$, we substitute the equilibrium effort level and exit rate (6) into $\dot{q}^u$. We want to identify the conditions on (q^u, q^g) such that $\dot{q}^u$ is positive. The derivative of $\dot{q}^u$ with respect to q^g is positive iff:

$$hq^u\lambda_b\left(\lambda_g(f + hr) - cr\right) > cf\lambda_g,$$

which is implied by the assumption that $q^u \geqslant F(1)$.[34] Therefore, $\dot{q}^u$ is positive iff:

$$q^g \geq \psi(q^u) := \frac{(h\lambda_b - c)(f(\beta+\lambda_b) + h\lambda_b q^u r)}{h\lambda_b(cr - \lambda_g(hr+f))\,(q^u)^2 + (h\lambda_b r(h\lambda_b - c) + c\lambda_g f)q^u + f(\beta+\lambda_b)(h\lambda_b - c)}. \tag{28}$$

□

Lemma B.3 (Properties of ψ). *The function ψ decreases in q^u for $q^u \in [F(1), 1]$. Moreover, $\psi(q^u) > F^{-1}(q^u)$ for all $q^u \in (F(1), 1]$, and $F^{-1}(q^u) = \psi(q^u)$ at $q^u = F(1)$.*

Proof. We want to argue that $\psi(q^u)$ decreases in q^u for $q^u \in [F(1), 1]$. Using algebra, one can verify that $\psi'(q^u) \leqslant 0$ iff $\nu(q^u) \leqslant 0$, with ν given by

$$\nu(q^u) := -h^2 r \lambda_b^2 \left(\lambda_g(f+hr) - cr\right)(q^u)^2 - 2fh\lambda_b\left(\lambda_b+\beta\right)\left(\lambda_g(f+hr) - cr\right)q^u + cf^2\lambda_g\left(\lambda_b+\beta\right).$$

This quadratic function ν is concave in q^u. Moverover, $\nu(0) > 0$ and $\nu(F(1)) \leqslant 0$, with strict inequality for $f > 0$. It follows that $\nu \leq 0$ for $q^u \in [F(1), 1]$, and hence that $\psi(q^u)$ decreases in q^u for $q^u \in [F(1), 1]$, strictly for $f > 0$.

Since $F(1) = \psi^{-1}(1)$, the two functions $F^{-1}(q^u)$ and $\psi(q^u)$ cross at $q^u = F(1)$. Moreover, for any $q^u \in (F(1), 1]$, it holds that:

$$F^{-1}(q^u) < \psi(q^u),$$

which can be verified using algebra.

□

Figure 6 illustrates the functions $F^{-1}(q^u)$ and $\psi(q^u)$. A necessary condition for a no-exit or a gradual-exit phase to exist is that $q^u \geqslant \frac{f}{\lambda_b h}$ and $q^g \geqslant F^{-1}(q^u)$. The dotted line corresponds to the constraint that $q^u \geqslant \frac{f}{\lambda_b h}$, and the solid curve to $q^g \geqslant F^{-1}(q^u)$. The dashed curve corresponds to $\psi(q^u)$, above which the belief q^u increases in t in the gradual-exit phase. The fact that $F^{-1}(q^u) < \psi(q^u)$ is intuitive. For (q^u, q^g) close to $(q^u, F^{-1}(q^u))$, the equilibrium exit rate is close to zero. This means that the belief q^u must go down. Therefore, $\psi(q^u)$ must lie above $F^{-1}(q^u)$.

Before we prove the result about feasible phase transitions, we establish the following lemma:

Lemma B.4. *The expected instantaneous effort $q^u(t)a^N(t)$ exerted by an uninformed player in the no-exit phase decreases over time.*

[34] Recall that $F(q^g)$ is decreasing in q^g.

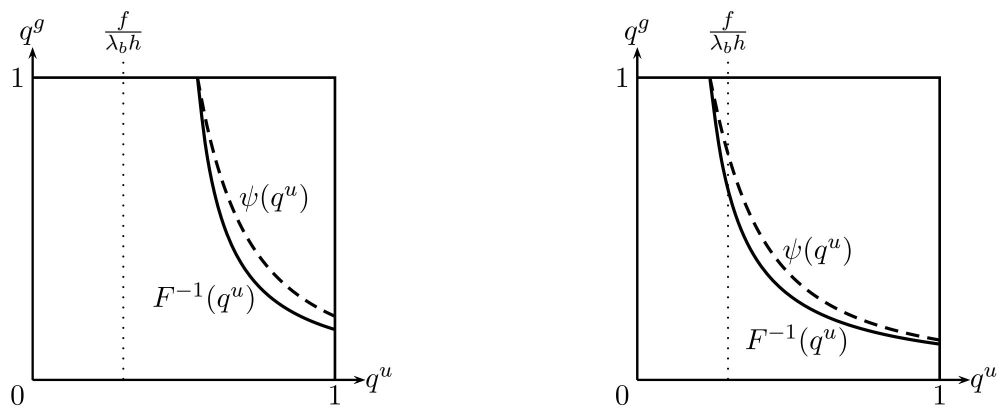


Figure 6: The F^{-1} and ψ functions for $(\lambda_g, \lambda_b, \beta, h, c, f) = \left(1, \frac{1}{3}, \frac{1}{3}, 1, \frac{2}{5}, \frac{1}{10}\right)$. The left-hand side corresponds to $r = \frac{1}{5}$. The right-hand side corresponds to $r = \frac{2}{3}$.

Proof. In the no-exit phase $q^u(t)$ strictly decreases in t, and $q^g(t)$ also decreases – strictly if $\beta < \lambda_g - \lambda_b$ (cf. Lemma 5.1).

By substituting $p^g(t),\ p^{bu}(t),\ p^{bi}(t)$ with $q^g(t),\ q^u(t)$, we write $q^u(t)a^N(t)$ as a function of $q^g(t),\ q^u(t)$:

$$q^u(t)a^N(t) = \frac{q^g(t)\left(\lambda_b(f+hr) + q^u(t)\left(cr - \lambda_g(f+hr)\right) - cr\right) - \lambda_b(f+hr) + cr}{c\lambda_b\left(q^g(t) - 1\right) - c\lambda_g q^g(t)}.$$

The partial derivatives of $q^u(t)a^N(t)$ are

$$\begin{aligned}\frac{\partial q^u a^N}{\partial q^u} &= \frac{q^g(t)\left(\lambda_g(f+hr) - cr\right)}{c\lambda_b\left(1 - q^g(t)\right) + c\lambda_g q^g(t)},\\ \frac{\partial q^u a^N}{\partial q^g} &= \frac{\lambda_b q^u(t)(\lambda_g(hr+f) - cr) - \lambda_g\left(\lambda_b(hr+f) - cr\right)}{c\left(\lambda_b\left(1 - q^g(t)\right) + \lambda_g q^g(t)\right)^2}.\end{aligned}$$

$\frac{\partial q^u a^N}{\partial q^u}$ is strictly positive given that $c < \lambda_g h$. Moreover, $\frac{\partial q^u a^N}{\partial q^g}$ is positive when $q^u(t)$ is sufficiently large, i.e., if

$$q^u(t) \geq \frac{\lambda_g\left(\lambda_b(f+hr) - cr\right)}{\lambda_b\left(\lambda_g(f+hr) - cr\right)}.$$

We next argue that the inequality above is always satisfied in the no-exit phase. Suppose $q^u(t)$ equals the right-hand side, then $q^u(t)a^N(t)\lambda_b h - f$ is strictly negative:

$$q^u(t)a^N(t)\lambda_b h - f = \frac{(f+hr)\left(h\lambda_b - c\right)}{c} < 0,$$

which yields a contradiction since $q^u(t)a^N(t)\lambda_b h$ must be above f in the no-exit phase. This completes the proof that $q^u(t)a^N(t)$ increases in $q^u(t), q^g(t)$, and hence decreases in t. $\square$

Proof of Lemma 5.2. First, we show that in equilibrium, play cannot transition from a no-exit to an immediate exit phase. To see this notice that in a no-exit phase, q^g weakly and q^u strictly decreases (cf. Lemma 5.1). Therefore, if the game transitions to another phase at time t, then $q^u < 1$ – the probability that ones opponent is informed is strictly positive. If the game were to transition to an immediate-exit phase, an uninformed player had no incentive to exert effort over $[t - \mathrm{d}t,\ t)$ since he expects to learn from the exit of an informed opponent at t. This then implies that an informed player has no incentive to stay over $[t - \mathrm{d}t,\ t)$, since an uninformed player exerts no effort. This shows that play cannot transition from a no-exit to an immediate-exit phase. It can only transition to a gradual-exit phase. The argument also shows that if there exists a time t at which the game transitions to the immediate-exit phase, then it must be that $q^u(t^-) = 1$.

We now argue that it is not feasible for the game to transition from a gradual-exit phase to the no-exit phase. To see this consider an informed agent. In the gradual-exit phase he is indifferent between exiting or not, whereas in a no-exit phase he strictly prefers to stay. The flow payoff of the outside option is exogenously given. Hence, given a belief-tuple (q^u, q^g) that satisfies the conditions in Lemma B.1, an informed player receives a higher flow payoff from being in a no-exit phase over being in a gradual-exit phase. Now suppose that there exists a time t such that the game transitions from a gradual-exit to a no-exit phase at t. Then right before the transition time, an informed player would strictly prefer to stay with the project and wait to enter the no-exit phase, instead of being indifferent between staying and exiting as required in a gradual-exit phase. In other words, there would exist some interval $[t - \mathrm{d}t,\ t)$, such that an informed player would strictly prefer to stay with the project – a contradiction to the game being in the gradual-exit phase during this time-interval.

A no-exit phase – and if $\lambda_g - \lambda_b < \beta$ also a gradual-exit phase – cannot last forever. Over time, the probability that the other player is informed increases (q^u decreases), and moreover the expected effort $q^u(t)a^N(t)$ of an uninformed player decreases over time (cf. Lemma B.4). The no-exit phase cannot persist beyond the time $\hat{t}$ at which abandoning the project becomes a better option, i.e., $q^u(t)a^N(t)\lambda_b h \leq f$. In the no-exit and gradual-exit phase, q^g is decreasing and eventually becomes so low that the necessary conditions for a finite-exit phase are not satisfied anymore (Lemma B.1).

Next, we argue that if play enters the immediate-exit phase, the game stays in this phase until all uninformed players exit at final-exit time t^I. By Lemma B.2, an immediate exit phase is possible only if $f \geq \min\{\lambda_b h,\ \bar{f}(q^g)\}$. But in this case, as we show in Proposition B.1, the only possible equilibrium phase is an immediate-exit phase. Hence, if play enters an immediate-exit phase, it stays there until the final-exit time.

Finally, to see that the game generically ends at a finite time, recall that, in the immediate-exit phase all players are uninformed, and become more pessimistic over time – strictly, if $\lambda_g - \lambda_b < \beta$. The immediate-exit phase can only last until the belief that the state is good drops to the level such that the marginal benefit from effort, $h\lambda^s(p^g)$, is exactly equal to the marginal cost c. Uninformed players are indifferent between all effort levels and, according to (9), choose the effort level at $a_j = f/c$ if efforts are interior. But then, for an uninformed player i – who benefits from his opponent's effort – this effort level generates a flow payoff at the same level as the outside option, that is, $a_j \cdot h\lambda^s(p^g) = f/c \cdot c = f$. Since for his own effort, marginal benefit equals marginal costs, it follows that all players exit at this time, $\bar{t}^I$, and take the outside option. If uninformed players exert full effort until the final exit-time, then players exit when the flow payoff from staying equals the outside option, $2h\lambda^s(p^g) - c = f$. Players are not willing to exert effort for lower beliefs, i.e., beyond this time. □

Proof of Corollary 5.1. In equilibrium, q^g is decreasing in all phases. As established in Lemma 5.2 if $\lambda_g - \lambda_b > \beta$ then play must eventually transition to an immediate-exit phase. If the belief drops to a level where the flow payoff from the project is equal to the outside option f, then all players will exit. No player has an incentive to stay beyond this point. The belief $q^g = p^g$ at this time $\bar{t}^I$ is given by:

$$a_i(\lambda^s(p^g(\bar{t}^I))h - c) + a_j\lambda^s(p^g(\bar{t}^I))h = 2h\lambda^s(p^g(\bar{t}^I)) \cdot a_j - c = f. \tag{29}$$

If efforts are interior, by evaluating (29) at (9), we obtain $p^g(\bar{t}^I) = \frac{c-h\lambda_b}{h(\lambda_g-\lambda_b)}$. If uninformed players exert full effort until the exit time, then $p^g(\bar{t}^I) = \frac{f+c-2h\lambda_b}{2h(\lambda_g-\lambda_b)} = p_2^*$ – the cooperative threshold. □

B.5 Proofs of Section 6

Lemma B.5. *A necessary condition for a three phase equilibrium to exist is for the project to exhibit a strong free-riding problem and players to be moderately patient.*

Proof of Lemma B.5. Suppose that the project exhibits a strong free-riding problem at $t = 0$, i.e., $p^g(0) \geq \bar{p}^I$ and $\lambda_b h \leq f$. The game must end with an immediate-exit phase, hence beliefs must eventually decrease to $p^g < \bar{p}^I$. This follows from Lemma B.2 and the observation that

$$f < \bar{f}(p^g) \quad \Leftrightarrow \quad p^g > \bar{p}^I.$$

Combining these considerations, we obtain that for a three-phase equilibrium to exist, it must be that $\overline{p}^I \in (0,1)$. Since $h\lambda_b < c < h\lambda_g$, $\overline{p}^I > 0$. Moreover, players being moderately patient (12) is equivalent to $\overline{p}^I < 1$.

Recall the definition of $\bar{f}(p^g)$ (22), which is strictly increasing in p^g. Note that

$$\bar{f}(1) = h\lambda_b \frac{r\,(h\lambda_g - c)}{\lambda_g\,(c - h\lambda_b)}.$$

Therefore, (23) – the necessary condition for the no-exit or gradual-exit phase to exist – holds if and only if there exists $p^g \in (0,1)$ such that

$$f < \min\left\{\lambda_b h, \bar{f}(p^g)\right\}.$$

Since $f < \bar{f}(p^g) \quad \Leftrightarrow \quad p^g > \overline{p}^I$, (23) holds if and only if the project exhibits a strong free-riding problem. □

Proof of Proposition 6.1 .

First note, that the necessary conditions of Lemma B.5 for a three-phase equilibrium to exist are satisfied. The project exhibits a strong free-riding problem at $t = 0$, that is, if and equilibrium exists it must start with a no-exit phase or gradual-exit phase. The latter is ruled out by observing that at $t = 0$, $(q^u, q^g) = (1, 0)$ and hence $q^g \geq \psi(q^u)$. By Lemma 5.1 q^u would be increasing in equilibrium – a contradiction. Combining this with the results of Lemma 5.2, we obtain that in equilibrium play must start with a no-exit phase, transition via a gradual-exit to an immediate-exit phase and end at a finite time. In section 4 it was established that

(1) it is optimal for players to exit immediately after a success or exit of the opponent,

(2) it is optimal for uninformed players to follow the strategies in Lemma 4.1 and Lemma 4.2 if the opponent does so.

Conditions for a *no-exit*, *gradual-exit* and *immediate-exit* phase to exist and hence the respective exit-behavior of informed players being optimal, were established in subsection B.3. Results on the motion of beliefs, feasible phase transitions, and the final-exit time $\overline{t}^I$ for the longest duration of experimentation were established in section 5.

All that is left to do is to determine the transition times t^N, t^G. We show that there exists a unique pair t^N, t^G such that, if the game proceeds to the gradual-exit phase at t^N, then the probability $p^{b,ui}(t)$ approaches zero at $t = t^G$. In other words, $q^u(t^G-) = 1$ is satisfied when the game transitions to the immediate exit phase (Lemma 5.2). We know moreover that, for informed players to be willing to exit immediately starting from t^G, it must be that $p^g(t^G) \le \overline{p}^I$, with $\overline{p}^I$ given by (11).[35] In other words, players must become sufficiently pessimistic such that the game turns into a weak free-riding problem.

During the no-exit phase, q^u is decreasing, and an informed player is willing to stay during the no-exit phase, if and only if $q^u(t)a_j^N(t)\lambda_b h \ge f$. Let $\hat{t}^N$ be the minimum time at which this inequality holds with equality. The transition time t^N must satisfy $t^N \in [0, \hat{t}^N]$.[36]

If $\exists t^G \in \left[t^N, \bar{t}^I\right)$, such that $\lim_{t\to t^G} q^u(t) = 1$, then it must be the case that $q^u(t)$ increases to 1 from below. This requires that $\exists\varepsilon > 0$ such that $\dot{q}^u(t) \ge 0$ for all $t \in \left(t^G - \varepsilon, t^G\right)$, i.e., $q^g \ge \psi(q^u)$ (Lemma 5.1). Substituting $q^u(t^G) = 1$ into $\psi(q^u)$ (28), we obtain that $q^g(t^G) \ge \overline{p}^I$. Recall the observation from above, that for informed players to be willing to exit immediately starting from t^G, it must be that $p^g(t^G) \le \overline{p}^I$. It follows that for such a t^G to exists, it must be the case that $q^g(t^G) = \overline{p}^I$, as well as, $q^u(t^G-) = 1$.

Putting together our observations and evaluating $\psi(q^u)$ (28), we obtain:

(i) The beliefs at time zero are $(q^u(t), q^g(t)) = (1,\ p^g(0))$. This point lies above ψ.

(ii) if the no-exit phase ended at $\hat{t}^N$, the beliefs at $\hat{t}^N$, $(q^u(\hat{t}^N),\ q^g(\hat{t}^N))$ would lie below ψ, that is, be such that $\dot{q}^u(\hat{t}^N)$ were strictly negative;

(iii) At the transition time t^G to the immediate-exit phase it must hold that $q^u(t^G-) = 1$ and $(q^u(t^G), q^g(t^G)) = (1,\ \overline{p}^I)$. This point is on the line $q^g = \psi(q^u)$.

As long as $(q^u,\ q^g)$ are above ψ , q^u is decreasing in the no-exit and increasing in the gradual-exit phase.

Lastly, the existence of t^N, t^G follows from continuity. To illustrate the continuity argument, consider figure 7. The dashed arrow shows the belief evolution in the no-exit phase. There exists $\hat{t} > 0$ such that if the game stayed in the no-exit phase from time 0 to time $\hat{t}$, the beliefs (q^u, q^g) at time $\hat{t}$ would cross the curve $q^g = \psi(q^u)$. If the game transitioned from the no-exit to the gradual-exit phase at $t = 0$, the beliefs during the gradual-exit phase would exit the gray area at point $t = 0$. If the game transitioned from the no-exit to the gradual-exit phase at $t = \hat{t}$, the beliefs during the gradual-exit phase would exit the gray area at point $t = \hat{t}$. By continuity, there exists $t^N \in \left(0, \hat{t}\right)$ such that if the game transitions from the no-exit to the gradual-exit phase at t^N, the beliefs during the gradual-exit phase will exit the gray area at beliefs $(q^u, q^g) = (1, \psi(1))$ at some t^G.

□

Proof of Proposition 6.2.

By assumption, the project exhibits a weak free riding problem, (11), which is a necessary

[35] Notice, that if $\lim_{t\to t^G} p^{b,ui}(t) \to 0$, $p^g(t^G) = \tilde{p}^g(t^G)$.

[36] In terms of beliefs, is must be that $q^u(t^N) \ge \hat{q}^u$ with $\hat{q}^u := \hat{q}^u(\hat{t}^N)$, the belief at time $\hat{t}^N$, which is zero for the full-effort case, and $\hat{q}^u = \frac{-(1-q^g)\lambda_b(f+hr)(h\lambda_b - c) + cfq^g\lambda_g}{hq^g\lambda_b(\lambda_g(f+hr) - cr)}$ if efforts are interior.

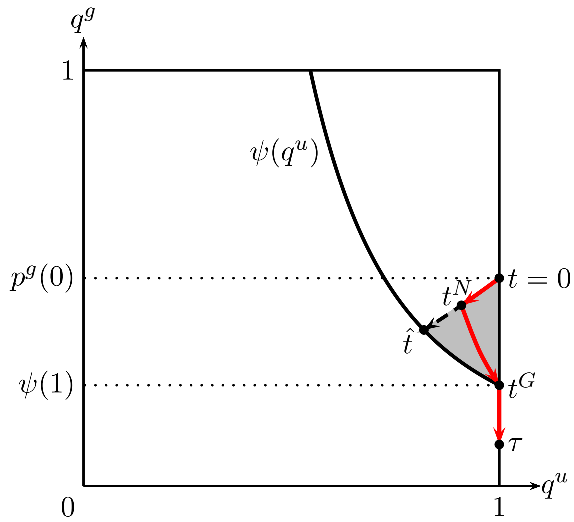


Figure 7: Belief evolution and transition times.

condition for informed players to exit immediately (Lemma B.2).[37]

In section 4 it was established that (1) it is optimal for players to exit immediately after a success or exit of the opponent, (2) it is optimal for uninformed players to follow the strategies in Lemma 4.2 if the opponent does so. Combining these facts with the results on the motion of beliefs in section 5 and the final-exit time $\bar{t}^I$ that was established as part of the proof of Lemma 5.2, yields existence of the immediate-exit equilibrium with the longest duration of experimentation. □

Proof of Corollary 6.1. At the latest final-exit time the marginal benefit from effort is equal to the marginal cost,

$$h\lambda^s(p^g)(\bar{t}^I) - c = f \quad \Leftrightarrow \lambda^s(p^g) = \frac{f+c}{2h}.$$

Thus, when $f \geqslant c$, an uninformed player's effort (9) right before the exit time $\bar{t}^I$ is:

$$\min\left\{\frac{r(h\frac{f+c}{2h} - c)}{c\lambda^{s,i}(p^g(\bar{t}^I))} + \frac{f}{c}, 1\right\} = 1.$$

Uninformed players exert full effort from the beginning until the exit time. At the exit time $\bar{t}^I$, the belief $p^g(\bar{t}^I)$ equals the cooperative threshold. This follows directly from Corollary 5.1 and since for $f \geqslant c$,

$$p_2^* = \frac{c+f-2h\lambda_b}{2h(la_g - \lambda_b} \geq \frac{c-h\lambda_b}{h(\lambda_g - \lambda_b)}.$$

[37]In Proposition B.1, we will show that the only possible equilibrium phase in this case is an immediate-exit phase.

Given that uninformed players exert full effort until they exit at the cooperative threshold, the cooperative solution is achieved as the equilibrium outcome for $f \geqslant c$. □

We have shown that if $f \geqslant \min\left\{\lambda_b h, \bar{f}(p^g(0))\right\}$, then there exists an immediate-exit equilibrium (Proposition 6.2). As the next proposition shows, if this condition holds, the only possible equilibrium phase is the immediate-exit phase.

Proposition B.1. *If $\lambda_g - \lambda_b > \beta$ and $f \geqslant \min\left\{\lambda_b h, \bar{f}(p^g(0))\right\}$, then the only possible equilibrium phase is the immediate-exit phase.*

Proof. If $f \geqslant \lambda_b h$, then an informed player always exits, so the only possible phase is the immediate-exit phase.

If $f \geqslant \bar{f}$, we argue that it is not possible to have a no-exit or a gradual-exit phase. Suppose $f \geqslant \bar{f}$, then $p^g(0) \leqslant \psi(1)$. At any time t in the no-exit or gradual-exit phase, it holds that $q^g \leqslant p^g(0)$ and $q^u \leqslant 1$. Moreover, ψ is strictly decreasing in q^u. It follows that (q^u, q^g) satisfies $q^g \leqslant p^g(0) \leqslant \psi(1) < \psi(q^u)$ for any $t > 0$. This implies that in a no-exit or gradual-exit phase q^u and q^g strictly decrease.

Now, suppose that there exists an equilibrium with a no-exit or a gradual-exit phase. Since q^u, q^g would be strictly decreasing in those phases, they cannot last forever, and moreover $q^u < 1$ for any $t > 0$. From Lemma 5.2, we know that uninformed players only exit from an immediate-exit phase, and that at the time of a transition to the immediate-exit phase, say at $\tilde{t}$, it must hold that $\lim_{t\to\tilde{t}} q^u(t) \to 1$ – a contradiction to $q^u < 1$ after a no-exit or gradual-exit phase.

This shows that no three-phase equilibrium can exist for the parameter range $f \in [\bar{f},\ \lambda_b h]$, which completes the proof. □

This means that we have an immediate-exit equilibrium iff $f \geqslant \min\left\{\lambda_b h, \bar{f}\right\}$.

B.6 Proofs of Section 7

Proof of Lemma 7.1. The effort level in the gradual-exit phase is given by $a^G(t) = \min\{\frac{f}{\lambda_b h q^u(t)},\ 1\}$ (cf Lemma 4.1). As discussed in Proposition 6.1, q^u is increasing in the gradual-exit phase on the equilibrium path. It follows that the effort level a^G decreases in t.

For the effort level in the immediate-exit phase (9) it holds that

$$\begin{aligned}\dot{a}^I(t) =& \frac{\partial a^I}{\partial p^g} \cdot \dot{p}^g(t)\\ =& \frac{r\left(\beta\lambda_g h + c(\lambda_g - (\lambda_b + \beta))\right)}{c \cdot \left(\lambda^{s,i}(p^g)\right)^2} \cdot \dot{p}^g(t)\end{aligned}$$

Given our assumption that $\lambda_g - \lambda_b > \beta$, the first factor is positive. Since $p^g = q^g$ is decreasing in the immediate-exit phase (cf. Lemma 5.1), it follows that a^I is decreasing in t. □

The following lemma is useful as a preliminary result. It characterizes how the effort level changes as a function of the beliefs q^g, q^u.

Lemma B.6. *The effort level in the no-exit phase weakly increases in q^g. It weakly increases in q^u if the markup of effort in the bad state is negative (i.e., $\lambda_b\left(h + \frac{f}{r}\right) \leqslant c$) and weakly decreases in q^u if it is positive.*

Proof of Lemma B.6. The effort in the no-exit phase is $a^N = \min\{a^N_{int},\ 1\}$, with

$$a^N_{int} := \frac{\lambda^s(p^g)(hr+f) - cr}{\lambda^u(p^g, p^{bu})c} = \frac{(f+hr)\left((1-q^g)\lambda_b + q^g q^u \lambda_g\right) - cr(1 - q^g(1-q^u))}{cq^u\left((1-q^g)\lambda_b + q^g\lambda_g\right)}, \quad (30)$$

where the last equality is obtained by substituting q^u, q^g.

Recall that throughout the no-exit phase it must hold that $q^u \geq \psi^{-1}(1)$. The derivative of a^N_{int} w.r.t. q^g is positive iff

$$cr\lambda_g > \lambda_b\left(cq^u r + (1-q^u)\lambda_g(f+hr)\right).$$

This inequality holds for any $q^u \in [\psi^{-1}(1), 1]$. Therefore, a^N_{int} increases in q^g, and hence so does $a^N = \min\{a^N_{int},\ 1\}$ (weakly).

The derivative of a^N_{int} w.r.t. q^u is positive iff

$$cr \geqslant \lambda_b(f+hr),$$

which completes the proof □

Proof of Proposition 7.1. If we show that a^N_{int} (30) increases in t, then $a^N = \min\{a^N_{int},\ 1\}$ also increases in t. Since both q^u and q^g decrease in the no-exit phase, it follows from Lemma B.6 that a^N_{int} decreases in t if $cr \geqslant \lambda_b(f+hr)$. This also implies that $cr < \lambda_b(f+hr)$ is a necessary condition for a^N_{int} to increase in t.

For the rest of the proof, we assume that $cr < \lambda_b(f+hr)$. We can obtain the derivatives $\dot{q}^u, \dot{q}^g$ during the no-exit phase by substituting $d_j = 0$ and $a_i = a_j$ given by (30) into (27). We then take the derivative of a^N_{int} w.r.t. time t and substitute $\dot{q}^u, \dot{q}^g$ into this derivative. This derivative $\dot{a}^N_{int}$ is positive iff $Y \cdot Z \geqslant 0$, with

$$Y = \left\{(f+hr)\left(\lambda_b - q^u\lambda_g\right) + c(q^u-1)r\right\}q^g + cr - \lambda_b(f+hr),$$
$$Z = z_1 q^g q^u + z_2 q^u + z_3 q^g + z_4,$$

with

$$\begin{aligned}
z_1 &= \lambda_b \left\{ (\lambda_g - \lambda_b)\,(\lambda_b(f+hr) - cr) + 2\,(\lambda - g - \lambda_b - \beta)\,(\lambda_g(f+hr) - cr) \right\}, \\
z_2 &= \lambda_b^2\,(\lambda_b(f+hr) - cr)\,, \\
z_3 &= \left\{ \lambda_b\,(\beta + \lambda_g) + \lambda_b^2 + \lambda_g\,(\beta - 2\lambda_g) \right\}(\lambda_b(f+hr) - cr)\,, \\
z_4 &= -\lambda_b\,(\lambda_b + \beta)\,(\lambda_b(f+hr) - cr)\,.
\end{aligned}$$

The derivative of Y w.r.t. q^g, $\frac{\partial Y}{\partial q^g}$, is linear in q^u. It is easy to verify that this derivative is negative for any $q^u \in [\psi^{-1}(1), 1]$. Hence, Y decreases in q^g. Since $Y|_{q^g=0} = cr - \lambda_b(f+hr) < 0$ when $q^g = 0$, Y is negative for all $q^g \in [0, 1]$.

It is easy to verify that $z_1 > 0$ and $z_2 > 0$ given that $cr < \lambda_b(f + hr)$. Therefore, Z increases in q^u. By substituting $q^u = 1$ into Z, we obtain that

$$Z|_{q^u=1} = (\lambda_b - \lambda_g)\left[\lambda_b(2cr + \beta(f+hr)) + cr\,(\beta - 2\lambda_g)\right] q^g - \beta\lambda_b\,(\lambda_b(f+hr) - cr)\,. \quad (31)$$

This term (31) increases in q^g iff

$$\beta \leqslant \frac{2cr\,(\lambda_g - \lambda_b)}{\lambda_b(f+hr) + cr}.$$

Under this condition, the highest Z is achieved when $q^u = 1$ and $q^g = 1$. This highest value $Z|_{q^u=1,q^g=1} \leq 0$ iff (13) holds.

Now suppose

$$\beta > \frac{2cr\,(\lambda_g - \lambda_b)}{\lambda_b(f+hr) + cr},$$

in which case $Z|_{q^u=1}$ given by (31) decreases in q^g. Under this condition, the highest Z is achieved when $q^u = 1$ and $q^g = 0$. This highest value $Z|_{q^u=1,q^g=0} \leq 0$ given the condition $cr < \lambda_b(f + hr)$. Therefore, we have shown that Z is negative when (13) holds. Putting all of these considerations together shows that $Y \cdot Z \geq 0$, and hence (30) increases in t when (13) holds. □

Proof of Proposition 7.2. We define the two state variables as in subsection B.2, $w_1(t) = e^{-\lambda_g \int_0^t \tilde{a}_i(s)\mathrm{d}s}$ and $w_2(t) = e^{-(\beta+\lambda_b)\int_0^t \tilde{a}_i(s)\mathrm{d}s}$. Let $\gamma_1(t)$ and $\gamma_2(t)$ be the corresponding co-state variables. Moreover, for convenience $x_1(t) = e^{-\lambda_g \int_0^t a_j(s)\mathrm{d}s}$ and $x_2(t) = e^{-(\beta+\lambda_b)\int_0^t a_j(s)\mathrm{d}s}$. The Hamiltonian of the problem is given by (15). Notice that the parameter region is such that only an immediate-exit phase can exist (**??**). Hence, the Hamiltonian is:

$$\begin{aligned}\mathcal{H}(\tilde{a}_i, w_1, w_2, \gamma_1, \gamma_2, t) =& e^{-rt}\left[p^g(0)w_1(t)x_1(t)(\tilde{a}_i(t)(h\lambda_g - c) + h\lambda_g a_j(t) - f)\right. \\ &\left. +(1-p^g(0))w_2(t)x_2(t)(\tilde{a}_i(t)(h\lambda_b - c) + h\lambda_b a_j(t) - f)\right] \\ &- \tilde{a}_i(t)\left[(\beta+\lambda_b)\gamma_2(t)w_2(t) + \lambda_g\gamma_1(t)w_1(t)\right].\end{aligned} \tag{32}$$

The belief $\tilde{p}^g(t)$ satisfies

$$\tilde{p}^g(t) = \tilde{p}^g(0)w_1(t)x_1(t) \tag{33}$$

Preliminary observation: Taking the derivative of $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ with respect to t and substituting $w_1'(t), w_2'(t), \gamma_1'(t), \gamma_2'(t)$, we obtain that the sign of $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ is the same as the sign of

$$a_j(t) - H(p^g(t)), \text{ where } H(p^g(t)) := \frac{r(h\lambda^s(p^g(t)) - c)}{c[\beta(1-p^g(t)) + \lambda^s(p^g(t))]} + \frac{f}{c}. \tag{34}$$

Notice that H increases in p^g.

Existence: Given the opponent j's exit time t_j^I and his effort level $\{a_j(t) : t \in [0, t_j^I)\}$, player i chooses the exit time t^I and the effort level before he exits $\{\tilde{a}_i(t) : t \in [0, t^I)\}$. The values of w_1, w_2 at time t^I are bounded from below. The lower bounds on the state variables are given by the control path with full effort throughout:

$$w_1(t^I) \geq \underline{w}_1(t^I) := e^{-\lambda_g t^I}, \quad w_2(t^I) \geq \underline{w}_2(t^I) := e^{-(\beta+\lambda_b)t^I}.$$

The Kuhn-Tucker conditions for the optimum then are:

$$\begin{aligned}\gamma_1(t^I) \geq 0, \quad \gamma_1(t^I)\left(w_1(t^I) - \underline{w}_1(t^I)\right) = 0 \\ \gamma_2(t^I) \geq 0, \quad \gamma_2(t^I)\left(w_2(t^I) - \underline{w}_2(t^I)\right) = 0.\end{aligned}$$

Case 1: We first examine the equilibrium such that $w_1(t^I) - \underline{w}_1(t^I) \geq 0, w_2(t^I) - \underline{w}_2(t^I) \geq 0$ do not bind. The Kuhn-Tucker conditions imply that $\gamma_1(t^I) = \gamma_2(t^I) = 0$. We need to consider two separate cases – the interior effort and the full effort case at the final-exit time.

Case 1.a: If $\tilde{a}_i$ is interior at t^{I-}, the derivative $\partial\mathcal{H}/\partial\tilde{a}_i$ equals zero at t^{I-}.[38] Substituting $\gamma_1(t^I) = \gamma_2(t^I) = 0$ into $\partial\mathcal{H}/\partial\tilde{a}_i(t^I) = 0$, we obtain that the posterior belief of state g at time t^I is

$$p^g(t^I) = \frac{c - \lambda_b h}{h(\lambda_g - \lambda_b)}.$$

[38]With slight abuse of notation, we use t^{I-} to denote the left limit of a function at t^I.

As discussed above, the sign of $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ is the same as the sign of $a_j(t) - H(p^g(t))$ (34). Moreover, $H(1) < 1$, since $r \le \frac{\lambda_g f(c-h\lambda_b)}{h\lambda_b(h\lambda_g - c)}$, and hence $H(p^g(t)) \in (0,1)$ for any $p^g(t)$ in $\left[\frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}, p_0\right]$.

In the equilibrium with the longest duration of experimentation, identified in Proposition 6.2, $a_j(t)$ is chosen such that $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ equals zero throughout. Then, the derivative $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ is also equal to zero throughout since $\partial\mathcal{H}/\partial\tilde{a}_i(t^I) = 0$. We now show that this is the unique equilibrium with the feature that the equilibrium effort right before players exit is interior.

Lemma B.7. *For the given parameter region (*TBA*), the equilibrium with the longest duration of experimentation, identified in Proposition 6.2, is the unique equilibrium with the feature that the equilibrium effort right before players exit is interior.*

Proof. First, at any time t, the effort level can take three possible values: (i) if $\partial\mathcal{H}/\partial\tilde{a}_i(t) > 0$, $\tilde{a}_i(t)$ equals one; (ii) if $\partial\mathcal{H}/\partial\tilde{a}_i(t) < 0$, $\tilde{a}_i(t)$ equals zero; (iii) if $\partial\mathcal{H}/\partial\tilde{a}_i(t) = 0$, the derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ equals zero so $\tilde{a}_i(t) = H(p^g(t))$. Now suppose effort is interior right before the final exit time, $\tilde{a}_i(t^I)^- = H(p^g(t)) < 1$ and (hence) $\partial\mathcal{H}/\partial\tilde{a}_i(t^I) = 0$. Let $t' \in \left(0, t^I\right)$ be the latest time at which $\tilde{a}_i(t') \neq H(p^g(t'))$. The effort level at time t' can be either (a) one or (b) zero.

(a) If $\tilde{a}_i(t') = 1$, then from (34) and since $H < 1$, it follows that $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t} > 0$ at t', implying that $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ is negative at t'. This contradicts the assumption that the effort level at t' is one.

(b) If the effort level at $\tilde{a}_i(t') = 0$, then by (34) $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t} < 0$ at t', implying that $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ is positive at t'. This contradicts the assumption that the effort level at t' is zero.

(a) and (b) show that there cannot be a latest time $t' \in \left(0, t^I\right)$ such that $\tilde{a}_i(t') \neq H(p^g(t'))$ and $a_i(t) = H(p^g(t))$ afterwards, i.e., for all $t \in \left(t', t^I\right]$. This completes the proof of the lemma. □

This completes the discussion of case 1.a.

Case 1.b: Now, suppose $\tilde{a}_i = 1$ is interior at t^{I-}, then $\partial\mathcal{H}/\partial\tilde{a}_i \ge 0$ at t^{I-}. Hence, the posterior belief of state g at time t^I is bounded from below:

$$p^g(t^I) \ge \frac{c - \lambda_b h}{h(\lambda_g - \lambda_b)}.$$

Moreover, it must hold that $p^g(t^I) \le p_1^*$. Otherwise, player i could improve by staying even if his opponent exits at time t^I. Now, pick any $p^g(t^I) \in (\frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}, p_1^*]$ as the belief at the exit time. For these beliefs, the first-order condition $\partial\mathcal{H}/\partial\tilde{a}_i$ is strictly positive at time t^I.

Moreover, the derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ is strictly positive since $a_j - H(p^g(t))$ is positive, and $a_i(t^I) = 1$. This implies that the FOC $\partial\mathcal{H}/\partial\tilde{a}_i$ decreases as we move back in time from t^I. At some time $\hat{t}$, the FOC $\partial\mathcal{H}/\partial\tilde{a}_i$ drops to zero.[39] The effort level is then given by $H(p^g(t))$ for $t \leq \hat{t}$. In other words, $a_i(t) = H(p^g(t))$ and $\partial\mathcal{H}/\partial\tilde{a}_i = 0$ for $t \in [0, \hat{t}]$ and $a_i(t) = 1$ and $\partial\mathcal{H}/\partial\tilde{a}_i > 0$ for $t \in (\hat{t}, t^I]$. An argument similar to the one presented in Lemma B.7 shows that the effort level cannot deviate from $H(p^g(t))$ before time $\hat{(t)}$. Therefore, the only possible pattern of the equilibrium effort level in this case 1.b is that there exists some $\hat{t} \in [0, t^I]$ such that equilibrium efforts are interior $a_i(t) = H(p^g(t))$ for $t \in [0, \hat{t})$, and $a_i(t) = 1$ for $t \in [\hat{t}, t^I]$.

Under the assumptions of this proposition, there always exists such an equilibrium in which players first exert interior effort and then exert full effort. Moreover, there always exists an equilibrium in which players exert interior effort equal to $H(p^g(t))$ throughout.

Case 2: We are left with characterizing the equilibrium such that $w_1(t^I) - \underline{w}_1(t^I) \geq 0$, $w_2(t^I) - \underline{w}_2(t^I) \geq 0$ bind. In this case, players exert full effort throughout. The FOC $\partial\mathcal{H}/\partial\tilde{a}_i(t^I) \geq 0$ implies that $p^g(t^I) > \frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}$. As discussed in case 1.b above, the belief at the exit-time, t^I must satisfy $p^g(t^I) < p_1^*$. Besides this restriction on the belief $p^g(t^I)$, the only constraint for such an equilibrium to exist is that $\partial\mathcal{H}/\partial\tilde{a}_i(t) \geq 0$ for any $t \in [0, t^I]$. Note that $\partial\mathcal{H}/\partial\tilde{a}_i(t^I)$ is the highest if $\gamma_1(t^I)$ and $\gamma_2(t^I)$ are zero. Moreover, the $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ does not depend on γ_1 or γ_2 (cf. (**??**)). Therefore, the condition $\partial\mathcal{H}/\partial\tilde{a}_i(t) \geq 0, \forall t \in [0, t^I]$ is easier to satisfy when we set $\gamma_1(t^I), \gamma_2(t^I)$ to be zero. In what follows, we discuss the conditions on the parameters for such an equilibrium to exit.

After substituting $\gamma_1(t^I) = \gamma_2(t^I) = 0$, the FOC at time t^I is:

$$\partial\mathcal{H}/\partial\tilde{a}_i(t^I) = e^{-rt^I}\left((1-p_0)e^{-2t^I(\beta+\lambda_b)}(h\lambda_b - c) + p_0 e^{-2\lambda_g t^I}(h\lambda_g - c)\right).$$

The derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ if players exert full effort throughout is given by (**??**), and we obtain an ODE with respect to $\partial\mathcal{H}/\partial\tilde{a}_i(t)$. Combining this ODE with the boundary condition at t^I, we solve for $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ explicitly. The condition that $\partial\mathcal{H}/\partial\tilde{a}_i(t) \geq 0, \forall t \in [0, t^I]$ is satisfied

[39] Recall, that parameters are such that in the equilibrium with the longest duration of experimentation efforts would be interior throughout, and hence $\partial\mathcal{H}/\partial\tilde{a}_i(0) = 0$.

if and only if $\partial\mathcal{H}/\partial\tilde{a}_i(t) \geq 0$ at time $t=0$, which gives the following condition on t^I:

$$\frac{(p_0-1)(\beta+\lambda_b)e^{t^I(-2(\beta+\lambda_b)-r)}(c-2h\lambda_b+f)}{2(\beta+\lambda_b)+r}+\frac{\lambda_g p_0 e^{t^I(-2\lambda_g-r)}(-c+2h\lambda_g-f)}{2\lambda_g+r} - \frac{(p_0-1)(\beta(c-f)+c(\lambda_b+r)-\lambda_b(hr+f))}{2(\beta+\lambda_b)+r}+\frac{p_0(\lambda_g(hr+f)-c(\lambda_g+r))}{2\lambda_g+r} \geq 0. \quad (35)$$

Since the left-hand side decreases in t^I, this condition imposes an upper bound on t^I. We denote this upper bound by $\overline{t^I}$. On the other hand, the belief at exit is at most p_1^*, so the following condition must be satisfied:

$$\frac{p_0 e^{-2\lambda_g t^I}}{(1-p_0)e^{-2t^I(\beta+\lambda_b)}+p_0 e^{-2\lambda_g t^I}} \leq \frac{c-h\lambda_b+f}{h(\lambda_g-\lambda_b)}. \quad (36)$$

This condition imposes a lower bound on t^I, denoted by $\underline{t}^I$. Such a full-effort equilibrium exists if and only if $\underline{t}^I \leq \bar{t}^I$. In this case, there exists a full-effort equilibrium for any $t^I \in [\underline{t^I}, \overline{t^I}]$.[40]

We have shown that given $p^g(t^I) > \frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}$, there exists some $\hat{t} \in \left[0, t^I\right]$ and an equilibrium in which players exert interior effort equal to $H(p^g(t))$ for $t \in \left[0, \hat{t}\right)$, and $a_i(t)=1$ for $t \in \left[\hat{t}, t^I\right]$. It must hold that either $\hat{t}>0$ or $\hat{t}=0$.

Uniqueness: We first consider the case of $\hat{t}>0$. For $t \in \left[0, t^I\right]$ the belief $p^g(t)$ satisfies the following condition: The belief $p^g(t)$ that the state is good conditional on no success, signal or exit by time t, satisfies (33). By the definition of w_1, w_2, we have $\log(w_1(t))/\log(w_2(t)) = \log(x_1(t))/\log(x_2(t)) = \lambda_g/(\lambda_b+\beta)$ for any $t \in [0, t^I]$. The three equations allow us to solve for $w_1(t)x_1(t), w_2(t)x_2(t)$ in terms of $p^g(t)$:

$$w_1(t)x_1(t) = \left(\frac{(1-p_0)p(t)}{p_0(1-p^g(t))}\right)^{-\frac{\lambda_g}{\beta+\lambda_b-\lambda_g}}, \quad w_2(t)x_2(t) = \left(\frac{(1-p_0)p(t)}{p_0(1-p^g(t))}\right)^{-\frac{\beta+\lambda_b}{\beta+\lambda_b-\lambda_g}}.$$

[40] There exists an example such that $\underline{t^I} < \overline{t^I}$. The parameters are $\lambda_g = h = 1, \lambda_b = 1/5, c = 1/4, r = 1/40, \beta = 1/10, p_0 = 1/2, f = 1/6$. If we set f to be $1/10$ instead of $1/6$, $\underline{t^I}$ is greater than $\overline{t^I}$, so no full-effort equilibria exist.

Substituting $w_1(t)x_1(t), w_2(t)x_2(t)$ and $\gamma_1(t^I) = \gamma_2(t^I) = 0$ into $\partial\mathcal{H}/\partial\tilde{a}_i(t^I)$, we obtain the FOC at time t^I:

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\tilde{a}_i(t^I)} =& e^{-rt^I} p_0(h\lambda_g - c)\left(\frac{(1-p_0)p^g(t^I)}{p_0(1-p^g(t^I))}\right)^{-\frac{\lambda_g}{\beta+\lambda_b-\lambda_g}} \\ &- e^{-rt^I}(1-p_0)(c-h\lambda_b)\left(\frac{(1-p_0)p^g(t^I)}{p_0(1-p^g(t^I))}\right)^{-\frac{\beta+\lambda_b}{\beta+\lambda_b-\lambda_g}}.\end{aligned}$$

This derivative is positive at t^I given that $p^g(t^I) > \frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}$. Since players exert full effort in $[\hat{t}, t^I)$, by (34), the derivative $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ must be weakly increasing for $t \in [\hat{t}, t^I)$ and equals zero at $t = \hat{t}$. Substituting $a_j(t) = 1$, $w_1(t)x_1(t) = e^{2\lambda_g(t^I-t)}w_1(t^I)x_1(t^I)$, and $w_2(t)x_2(t) = e^{2(\lambda_b+\beta)(t^I-t)}w_2(t^I)x_2(t^I)$ into $\frac{\mathrm{d}(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\mathrm{d}t}$, we obtain the value of $\frac{\mathrm{d}(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\mathrm{d}t}$ for $t \in [\hat{t}, t^I)$:

$$\begin{aligned}\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t} =& p_0 w_1(t^I)x_1(t^I)e^{\lambda_g(t^I-t)-rt}(c(\lambda_g+r) - \lambda_g(hr+f)) \\ &+ (1-p_0)w_2(t^I)x_2(t^I)e^{(\beta+\lambda_b)(t^I-t)-rt}(\beta(c-f) + c(\lambda_b+r) - \lambda_b(hr+f)).\end{aligned}$$

This derivative, along with the value of $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ at $t = t^I$, allows us to solve for $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ for any $t \in [\hat{t}, t^I)$. The first condition that $\hat{t}, t^I$ must satisfy is that (1) $\partial\mathcal{H}/\partial\tilde{a}_i(t) = 0$ for $t = \hat{t}$.

When $t \in [0, \hat{t})$, the effort level equals $H(p^g(t))$ as defined in (34). This, combined with the initial condition $w_1(0) = 1$, allows us to solve for $w_1(t)$ for any $t \in [0, \hat{t})$. At the same time, (2) $w_1(\hat{t})x_1(\hat{t}) = e^{\lambda_g(t^I-\hat{t})}w_1(t^I)x_1(t^I)$. The two conditions (1) and (2) allow us to solve for $\hat{t}$, t^I in terms of $p^g(t^I)$.

We next examine the case that $\hat{t} = 0$. Both players exert full effort. The value of t^I is determined by the belief $p^g(t^I)$ at time t^I. Note that such a full-effort equilibrium with the length t^I exists if and only if both (35) and (36) are satisfied.

Next, we want to argue that the two cases $\hat{t} > 0$ and $\hat{t} = 0$ are exclusive. Suppose not. Suppose, for some given belief of state g at the exit time, there exists a full-effort equilibrium with length t_1^I and an equilibrium with interior effort in $[0, \hat{t}_2)$ and full effort in $[\hat{t}_2, t_2^I)$. We refer to the latter as the interior-effort equilibrium. Then, it must hold that $t_2^I > t_1^I$ and $t_2^I - \hat{t}_2 < t_1^I$. For a fixed $\tau \le t_2^I - \hat{t}_2$, the derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ at $t = t_1^I - \tau$ for the full-effort equilibrium is greater than the derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ at $t = t_2^I - \tau$ for the interior-effort equilibrium due to the condition that $t_1^I - \tau < t_2^I - \tau$ and the multiplier e^{-rt} in the derivative. Therefore, it is not possible for $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ to reach zero at time $\hat{t}_2$ in the interior-effort equilibrium, while the derivative $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ is weakly positive at time 0 in the full-effort equilibrium. This contradiction shows that the two cases $\hat{t} > 0$ and $\hat{t} = 0$ are exclusive.

Lastly, we want to show that for some given belief of state g at the exit time, there cannot be two interior-effort equilibria characterized by $(\hat{t}_2, t_2^I)$ and $(\hat{t}_3, t_3^I)$. Suppose not. Without loss, suppose that $\hat{t}_3 < \hat{t}_2$. Then, it must hold that $t_3^I < t_2^I$ and $t_3^I - \hat{t}_3 > t_2^I - \hat{t}_2$. For any $\tau < t_2^I - \hat{t}_2$, given that $t_3^I < t_2^I$, the derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ at $t = t_3^I - \tau$ in the $(\hat{t}_3, t_3^I)$ equilibrium is larger than the derivative $\frac{\partial(\partial\mathcal{H}/\partial\tilde{a}_i(t))}{\partial t}$ at $t = t_2^I - \tau$ in the $(\hat{t}_2, t_2^I)$ equilibrium. Therefore, it cannot be true that it takes a longer length of $t_3^I - \hat{t}_3$ for the derivative $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ to reduce to zero as we move down from t_3^I while it takes a shorter length of $t_3^I - \hat{t}_3$ for the same derivative $\partial\mathcal{H}/\partial\tilde{a}_i(t)$ to reduce to zero. This is a contradiction.

This completes the proof that there exists a unique tupel $(\hat{t}, t^I)$ for any given exit belief $p^g \in \left(\frac{c-\lambda_b h}{h(\lambda_g-\lambda_b)}, p_1^*\right]$.

□

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