Equity ATMs*

Nicolas Inostroza[†] Anne-Katrin Roesler[‡] Anton Tsoy[§]

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Abstract

Small high-growth companies often sell common shares to cover repeated cash needs. This is surprising, as the classical theory predicts debt in such environments. We present a model, in which the firm's owner in anticipation of several liquidity shocks designs securities to sell to outsiders and her private signal about firm's cash flows. We show that it is optimal to use common equity as an ATM: as shocks arrive, the owner covers them by selling common shares. Under an optimal signal, selling common shares does not hurt liquidity of future share issues, which is generally not true for other securities.

Keywords: security design, asymmetric information, information design, equity, preferred stock

JEL Classification: D82, D86, G32

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[†]University of Toronto and FTG. Email: nicolas.inostroza@rotman.utoronto.ca

[‡]University of Toronto. Email: ak.roesler@utoronto.ca.

[§]University of Toronto and FTG. Email: anton.tsoy@utoronto.ca.

1 Introduction

There is value locked in a company's potential to generate future cash flows. However, immediate cash needs often arise, such as R&D expenditures, working capital requirements, interest payments on floating-rate loans, legal expenses from consumer lawsuits, or regulatory fines. The basic principle of finance asserts that one way to unlock this value is by selling shares to external investors. This process can be repeated as long as sufficient intrinsic value remains. Metaphorically, companies can use equity as a source of liquidity, similar to how ATMs were used for cash withdrawals. We refer to this practice as "equity ATMs."

Companies use equity as ATMs in practice. Frank and Goyal (2003), Leary and Roberts (2010) document prevalence of external equity financing, in particular, among small high-growth firms. Such financing is often done through seasoned equity offerings (SEOs) or private investment in pubic equity (PIPE) offerings. DeAngelo et al. (2010) and Floros and Sapp (2012) show that repeated equity offering are common and cannot be explained by the traditional pecking order, trade-off, or market timing theories, but rather, they better fit the "need for cash" hypothesis, stating that companies issue common shares in response to immediate cash needs.

From the standpoint of the classical theory, the use of highly informationally sensitive common equity is surprising in such circumstances. Apart from the frictionless benchmark (and few papers reviewed below), this is not what the classical corporate finance theory predicts. As long as there are frictions, such as adverse selection or moral hazard, the classical literature does not prescribe equity issuance as an optimal response to liquidity shocks. Rather, the optimal security is usually debt or has debt-like features. The question is: what makes common equity special among the multiplicity of possible securities? Further, the literature often focuses on a single round of security issuances, which does not allow for an explanation of repeated issuance of equity.

We present a parsimonious model of security design in which common equity is optimal, in line with the basic finance principles and empirical evidence, and it is used as an ATM to cover multiple upcoming liquidity shocks. In our model, a firm owner anticipates a series of liquidity shocks that occur before the firm's cash flows are realized. These shocks in total will not bankrupt the firm so there is enough value locked-in to cover all shocks. The owner can cover each shock by selling securities backed by future cash flows to a monopolistic liquidity supplier (distinct for each shock). The monopolistic position of liquidity suppliers reflects that equity offerings are intermediated by lead underwriters with significant market power (who negotiate the price on behalf of other investors) and are often sold to a handful of sophisticated institutional investors (especially, in the case of PIPE offerings). It also reflects the issuer's urgent need for cash.

There are two market frictions in the model. First, there is asymmetric information when securities are sold. When the shocks hit, the owner gets a private signal about future cash flows so she is better informed about the value of securities than liquidity suppliers. Second, there is an agency friction as in Jensen (1986). The firm is governed by the opportunistic management who can divert for private benefits any amount of money raised in excess of what is needed to cover liquidity shocks. The latter implies that the owner cannot stock-pile cash (without it being diverted for private benefits by the management), and instead, has to raise with each security sale just enough money to cover each new shock.

Before receiving any private information, the owner designs a sequence of securities to accommodate the upcoming liquidity shocks. This captures the common practice of shelf-registration when issuers pre-register securities with regulators to be able to quickly react to changing economic environments and avoid costly regulatory delays. We allow for a rich class of securities that satisfy limited liability (i..e., each security's payoff is positive and does not exceed realized cash flows) and double monotonicity (i.e., each security's payoff as well as the payoff of the security retained by the owner is weakly increasing in realized cash flows). The novel element of the model is that the owner can also design information, namely, her signal about cash flows that she will receive in the future (before trading the securities). We allow for maximal freedom and suppose that the owner can design any Bayesian signal about underlying cash flows.

Our main result is that it is optimal for the owner to sell common shares in the firm to the liquidity suppliers. In our model, common shares represent a fraction of the levered equity, which captures two characteristic properties of common equity: (i) they are junior to other claims, e.g., debt or preferred equity; (ii) they represent a claim on the firm's cash flows after all senior claims are paid in full (and only in this case). Importantly, in the optimum, the owner uses common equity as an ATM: as another liquidity shock arrives, she sells a share of common equity to compensate for it. On the information design part, the owner designs her signal so that to ensure that common shares can always be sold to liquidity suppliers (without any inefficient screening) while maximizing the share price.

What makes common equity optimal? The distinctive property of common equity is its linearity in cash flows (post other senior payments). This linearity is valuable when the owner can optimally design the information. Specifically, due to linearity, a signal that is optimal to sell 1% of common equity is also optimal for selling 5%, 20%, 100%, or any other share. Thus, selling shares to cover the first liquidity shock does not hurt the owner's ability to sell future shares to accommodate subsequent shocks. This is generally not true for non-linear security. For example, selling a debt security might require the owner to learn more information about the downside, which might be not valuable when selling the residual levered equity security. Generally, the owner prefers to learn different information about these two securities, which forces the owner to prioritize one security over the other in the information design.

Optimality of common equity exhibits remarkable robustness. It does not matter whether the shocks are known in advance or are uncertain, whether liquidity suppliers observe each others' trades or not, whether trading in securities is sequential or simultaneous. In turn, all these factors are potentially important for selling other non-linear securities. This highlights the special role that common equity plays in addressing liquidity needs of companies.

Our theory presents a novel role of equity in financing. As mentioned above, sales of equity in response to liquidity shortages are documented empirically. A typical equity issuer is a small company with growth options that uses the proceeds to cover persistent cash needs, such as R&D expenses. Our theory fits well with this empirical evidence.

The benefit of common equity in raising cash for repeated cash flow shocks seems to be missing from the quantitive exploration of firms' financing. DeAngelo et al. (2011) estimate a structural model of repeated financing. In line with the pecking order theory, they suppose that equity issuance is costly compared to debt financing due to its informational sensitivity. The estimated model fits remarkably well all the moments in the data, apart from the equity issuance, which is underestimated. DeAngelo et al. (2011) shows that adding debt costs (equivalently, extra benefits of equity issuance) would improve the model fit. Our theory provides a microfoundation for a novel benefit of equity over other securities.

Literature Review This paper contributes to the vast literature on security design under asymmetric information starting from Leland and Pyle (1977), Myers and Majluf (1984), and Myers (1984). In this literature, debt or debt-like securities often arise as optimal when there are frictions, such as adverse selection or moral hazard (Innes 1990, Nachman and Noe 1994, DeMarzo 2005, DeMarzo, Kremer and Skrzypacz 2005, Dang, Gorton and Holmström 2013, Daley, Green and Vanasco 2020, Li 2022, Figueroa and Inostroza 2023, Asriyan and Vanasco Forthcoming, Gershkov, Moldovanu, Strack and Zhang 2023). Boot and Thakor (1993), Fulghieri and Lukin (2001), Axelson (2007), Yang and Zeng (2019), Fulghieri et al. (2020), Daley et al. (2020) show securities, such as equity or levered equity, can be preferred to less informationally sensitive securities, because they stimulate information acquisition, improve aggregation of private information, and better exploit complementarity with public information sources. Few paper study design of multiple securities offered in response to different liquidity shocks or sold to different investors. This literature restricts attention to only debt and equity and focuses on the trade-off between these two securities, especially when it comes to studying dynamics of financing (see, e.g., Hennessy et al. 2010, Morellec and Schürhoff 2011, DeAngelo et al. 2011).

Our contribution to this literature is in showing that repeated sales of common shares are optimal within universe of securities in response to repeated liquidity shocks. Generally, this result is hard to obtain in the classical approach that assumes exogenous signals. One reason is that, in a static security design problem, the solution often has a bang-bang property that the slope of the optimal security is either one or zero leading to optimality of debt or levered equity (Nachman and Noe (1994), Biais and Mariotti (2005), Fulghieri et al. (2020)). For this reason, linear securities, such as shares of common equity, are generally suboptimal. On top of that, considering the design of many securities further complicates the problem due to the dynamics issues, such as time-inconsistency.

In contrast, our approach is to consider a rich class of signals, and following the information design literature, focus on the optimal signal design. This approach allows us to go beyond simple perfectly revealing signals and greatly simplifies the problem. We get a simple and natural solution with a novel economic intuition. Selling common shares is optimal when the owner anticipates future shocks, because it does not compromise the owner's future ability to raise funds.

The common equity represents a linear security – it is a share of cash flows (after payments to other senior securities). In this respect, our paper is related to the literature on optimality of linear contracts in the moral hazard (Carroll 2015, Walton and Carroll 2022) and adverse selection (Admati and Pfleiderer 1994, Malenko and Tsoy 2020) models. Differently from these papers that stress Knightian uncertainty or robustness features of equity, we obtain linear contracts in a fully Bayesian model. Our mechanism is quite different from those paper and relies on the optimal (rather than worst-case) information design. Distinctly from this literature which primarily studies static models, we show optimality of repeated issuance of levered equity. In the robustness literature, the sufficient conditions for optimality of linear contracts in the dynamic setting derived by Liu (2022) are quite stringent, which is in contrast to the general optimality of them in our setup.

We contribute to the literature on optimal information design (Bergemann, Brooks and Morris 2015, Roesler and Szentes 2017, Glode, Opp and Zhang 2018, Kartik and Zhong 2023). Within this literature, Theorem 1 is related to the optimality of bundling in the presence of information design in Deb and Roesler (2024). While related, our result is different in that both the information design and the decision to bundle or not several securities is made by the owner, while in Deb and Roesler (2024) those decisions are with different parties.

The paper is organized as follows. Section 2 presents the model. Section 3 analyzes the escrow problem, which is of interest on its own. Section 4 presents our main result.

Section 5 discusses extensions and robustness. All omitted proofs are in Appendix and Online Appendix.

2 Model

We present a model of the optimal firm's response to repeated liquidity shocks. While the model is parsimonious, we use the application to repeated security issues by small high-growth firms to motivate our assumptions.

There are three stages: $t = 0$ (design stage), $t = 1$ (trading stage), $t = 2$ (payout stage). The firm's asset generates cash flows X at $t = 2$ distributed according to the CDF H on a positive support $\mathcal{X}$ with $\underline{x} > 0$ and $\overline{x} < \infty$ being the minimal and maximal elements in $\mathcal{X}$, respectively. The trading stage $t = 1$ consists of I rounds indexed by $i = \overline{1, I}$. In each round i, the firm is hit by a liquidity shock and must pay $R_i > 0$ to continue operations (e.g., R&D expenses, changes in working capital, increased interest payments, consumer lawsuit expenses, regulatory fines). If the firm fails to cover the liquidity shock in round i, then it immediately ceases operations in this round and the asset is destroyed with no cash flows at $t = 2$ or recovery value at $t = 1$. In the baseline model, the sequence of shocks $\boldsymbol{R} = (R_i)_{i=1}^{I}$ is common knowledge at $t = 0$. Section 5 considers uncertainty about shocks.

There is a firm's owner and a sequence of liquidity suppliers – one for each trading round i. All agents are risk-neutral. At $t = 1$, liquidity suppliers value future cash flows at X, while the owner values them at $\delta X, \delta \in (0, 1)$. A lower discount factor can be due to other investment opportunities available to the owner, her own liquidity needs, or regulatory costs of holding risky investment. It can also capture the owner's costs of under-diversification, because of the excessive exposure to the idiosyncratic firm risk.

Apart from cash flows X, the owner also gets private benefits B if the firm survives until $t = 2$ (irrespective of how cash flows are allocated), which cannot be pledged to liquidity suppliers at the trading stage. For example, B can capture the owner's private benefits of control. Alternatively, B is the value of assets pledged as collateral for pre-existing debt that is lost if the firm is liquidated.

We suppose that the owner has deep pockets at $t = 1$ and has an option to pay R_is herself. For example, the owner can inject equity into the firm to cover the liquidity shocks. Alternatively, the firm has liquid assets that it can sell to cover liquidity shocks. Yet, unlike the cash flows from the firm's asset, the owner and the liquidity suppliers value equally these liquid assets. We assume that $B \geq R \equiv \sum_{i=1}^{I} R_i$, and so, the owner always prefers to continue operations even if it entails her equity injection or selling liquid assets. This assumption allows us to abstract from signaling considerations, in which the owner's decision to partially contribute to R_i serves as a credible signal of the asset quality and reveals information about cash flows to liquidity suppliers.[1]

Because of differences in discount factors, it is efficient to raise external funds from liquidity suppliers by pledging future cash flows X rather than cover the liquidity shocks herself. To realize these gains from trade, at the design stage $t = 0$, the owner designs securities $F_i = \varphi_i(X), i = \overline{1, I}$, whose payoff is contingent on X, to be traded in each round i. Henceforth, we use the payoff function φ_i to refer to the security offered in round i.

The assumption that securities are designed well in advance of liquidity shocks is motivated by the common practice of *shelf registration*, in which companies pre-file registration documents for a series of securities. This way, they can avoid regulatory delays and can quickly issue securities to fulfill future liquidity needs, which is particularly valuable when liquidity needs are urgent.[2] In practice, securities are not contingent on shocks, but rather companies only specify the class of securities to be issued and the total amount to be raised. As we show below, this feature arises endogeneously as part of the optimum.

The trade of securities at $t = 1$ is inhibited by two frictions. First, at the beginning of the trading stage, the owner has superior information about future cash flows. She receives a private signal S about X. In each trading round i, only liquidity supplier i is present and she has all the bargaining power. The monopolistic position of liquidity suppliers reflects situations when the firm needs to raise liquidity in times of scarce liquidity, such as, during the crisis periods when other firms also experience liquidity shocks. It also captures the decentralized nature of the market for financing. An investor willing to buy securities today might not be there to supply liquidity tomorrow because of other investment opportunities or her own liquidity needs. This, in particular, prevents any long-term contracting between the owner and liquidity suppliers. In terms of practical relevance, our model speaks to how companies raise liquidity from financial markets through secondary public offerings of securities, private security placements, spin-offs of divisions, but is less relevant for long-term relationship financing through banks.

For now, we suppose that the trading outcomes between the owner and each liquidity suppliers are not observable by any other market participants. We relax this assumption in Section 5. For any security F_i, the liquidity supplier i offers a posted price P_i at which he is

[1]Nachman and Noe (1994), Fulghieri et al. (2020), Malenko and Tsoy (2020) study signaling with securities. By assuming large private benefits and "shutting down" signaling, we insulate the role of information design. It is interesting to explore how security and information design interacts with subsequent signaling motives of the owner, which we leave for future studies.

[2]In practice, certain equity offerings to qualified institutional investors are exempt from immediate registration upon issuance. This way, not including other securities into the shelf registration effectively commits the firm to issue equity to a restricted set of institutional investors, as at the moment when the liquidity shock arrives, other forms of financing are associated with additional regulatory costs and delays.

willing to buy it. The restriction to posted prices is without loss of optimality for liquidity suppliers (by Proposition 1 in Biais and Mariotti 2005).[3]

Second, there is a standard agency friction on the firm side (Jensen 1986). We suppose that, when the firm raises P_i in round i, unless P_i is spent by the firm immediately to compensate the liquidity shock, it would be completely appropriated by the firm's management in the form of private benefits and none of the benefits would accrue to the owner or security holders. Thus, the owner gets $\min\{P_i, R_i\}$ from the security sale. This implies that the firm cannot raise all R_is only once, and instead, needs to raise money in each round.

Remark 1. *Since in the baseline model trading rounds are informationally insulated from each other, our model has an alternative interpretation. We can suppose that, instead of a sequence of liquidity shocks, there is a single liquidity shock of size R at stage $t = 1$, and liquidity suppliers indexed by $i = \overline{1, I}$ each willing to commit only up to R_i to the purchase of the firm's securities, which again leads to the owner getting at most R_i from the security sale to liquidity supplier i. As in the baseline model, we assume that that all security placement negotiations happen in private and their outcomes are non-contractible. In practice, security offerings are often made to a syndicate of institutional investors who tend to have mandates limiting their participation in security issuances of any single firm. Because of this portfolio constraint, outside investors do not compete with each other for securities, as each one of them covers only a part of the firm's liquidity needs. This way, outside investors have all the market power and simultaneously make take-it-or-leave-it price offers to the owner. The private nature of negotiations preclude information leakages and signaling through security offerings.*

At the design stage $t = 0$, the owner optimally designs the signal S about X that she privately learns at $t = 1$ and the sequence of securities $\boldsymbol{\varphi} \equiv (\varphi_i)_{i=1}^I$ to be sold in each trading round i. We suppose that the owner issues a single security in each round.[4]

We suppose that securities $(\varphi_i)_{i=1}^I$ satisfy (i) limited liability, i.e., each $\varphi_i(x) \in [0, x]$ for all x; (ii) monotonicity, i.e., each $\varphi_i(x)$ is weakly increasing in x; (iii) double monotonicity, i.e., $x - \sum_{i=1}^I \varphi_i(x)$ is non-negative and weakly increasing in x. Let $\mathbf{\Phi}$ be the set of all such sequences $\boldsymbol{\varphi}$s. This class of securities includes all securities commonly used in practice. It naturally generalizes to multiple securities the limited liability and monotonicity assumptions that are standard in the security design literature (Nachman and Noe 1994, DeMarzo and Duffie 1999, Biais and Mariotti 2005), and coincides with that used recently in Asriyan and Vanasco (Forthcoming). The monotonicity assumption of securities can be motivated by the

[3]Biais and Mariotti (2005) additionally assume H admits a density on $[\underline{x}, \overline{x}]$. While required to prove other results in their paper, this assumption is not used in their proof of Proposition 1, which holds more generally.

[4]As follows from Theorem 1 below, this assumption is without loss of optimality.

standard "sabotage" argument that a party owning a non-monotone security has perverse incentives to destroy cash flows in order to increase her own payout.

At $t = 0$, the owner also designs a signal S about X, which she privately learns at the beginning of $t = 1$. A signal S is described by the probability space $(\mathcal{X} \times \mathcal{S}, \mathscr{X} \times \mathscr{S}, \nu)$, where $\mathcal{S}$ is a sufficiently rich Polish space of possible signal realizations (in particular, $\mathcal{X} \subseteq \mathcal{S}$) and $\nu(x, s)$ is the probability measure on the product σ-algebra $\mathscr{X} \times \mathscr{S}$. The marginal distribution of ν on $\mathcal{X}$ must coincide with the prior distribution of X, H. We denote by G^X the CDF of the conditional expectation $\mathbb{E}[X|S]$ induced by ν, and by $\mathcal{G}^X$ the set of all G^X induced by some signal S.

For any security φ_i, let G^{φ_i} be the CDF of $\mathbb{E}[\varphi_i(X)|S]$ and $\mathcal{G}^{\varphi_i}$ be the set of all such G^{φ_i}s. For any sequence of securities $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$, let $\boldsymbol{G}^{\boldsymbol{\varphi}} = (G^{\varphi_1}, \ldots, G^{\varphi_I})$ be the CDFs of conditional expectations of each security $\mathbb{E}[\varphi_i(X)|S]$ and $\boldsymbol{\mathcal{G}}^{\boldsymbol{\varphi}}$ be the set of all such $\boldsymbol{G}^{\boldsymbol{\varphi}}$s.[5] Clearly,

$$\boldsymbol{\mathcal{G}}^{\boldsymbol{\varphi}} \subseteq \mathcal{G}^{\varphi_1} \times \cdots \times \mathcal{G}^{\varphi_I}. \tag{1}$$

but the inclusion is in general strict whenever $I > 1$. Intuitively, learning information about security φ_i by observing some signal S about X forces the owner to also learn information about securities $\varphi_j, j \neq i$. This means that there are distributions in $\mathcal{G}^{\varphi_1} \times \cdots \times \mathcal{G}^{\varphi_I}$, which cannot be obtained by observing a signal about X.

2.1 Security Design Problem

We next formulate the security design problem at date $t = 0$.

We first compute the owner's payoff from trading security φ_i in round i. It is useful to first suppose that the agency friction is not binding in round i, i.e., the liquidity supplier i offers $P_i \leq R_i$. Since trading outcomes in all rounds are private, the owner has no signaling motives and she accepts P_i if and only if $\delta\mathbb{E}[\varphi_i(X)|S = s] \leq P_i$. Given the distribution of the expected security payoffs G^{φ_i}, the liquidity supplier i's payoff when offering a price P_i for security φ_i is given by

$$\pi(P_i|G^{\varphi_i}) = \int_{-\infty}^{P_i/\delta} (z - P_i)\,\mathrm{d}G^{\varphi_i}(z),$$

which has a solution

$$P(G^{\varphi_i}) \equiv \sup\left\{\arg\max_{P_i} \pi(P_i|G^{\varphi_i})\right\}. \tag{2}$$

Here, we assume that, when multiple prices are optimal, the owner and liquidity supplier i

[5] Throughout the paper, we use bold letters to denote sequences.

coordinate on the owner's most preferred price.

If $P(G^{\varphi_i}) \leq R_i$, then the agency friction is indeed not binding and the owner's ex-ante expected payoff from trading φ_i is given by

$$V(G^{\varphi_i}) \equiv \int_{-\infty}^{P(G^{\varphi_i})/\delta} (P(G^{\varphi_i}) - \delta z)\, \mathrm{d}G^{\varphi_i}(z). \tag{3}$$

If $P(G^{\varphi_i}) > R_i$, then the agency friction comes into play. For any offer $P_i > R_i$, the excess $P_i - R_i$ is appropriated by the firm's management and the owner's payoff is only R_i (her saving from not contributing to shock i herself). Because of this, the owner accepts $P_i > R_i$ if and only if $\delta\mathbb{E}[\varphi_i(X)|S = s] \leq R_i$, and so, for the liquidity supplier i, any $P_i > R_i$ is strictly dominated by $P_i = R_i$. In this case, the owner gets

$$V^{R_i}(G^{\varphi_i}) \equiv \int_{-\infty}^{R_i/\delta} (R_i - \delta z)\, \mathrm{d}G^{\varphi_i}(z). \tag{4}$$

Lemma 3 in the Appendix shows that it is without loss of optimality for the owner to focus on securities φ_i and distributions G^{φ_i} such that $P(G^{\varphi_i}) \leq R_i$. Thus, the agency friction boils down to the constraint

$$P(G^{\varphi_i}) \leq R_i, i = \overline{1, I}. \tag{5}$$

Notice that the security φ_i affects the owner's and liquidity supplier i's payoffs only through G^{φ_i}, but otherwise does not appear in the expressions above.

The owner's expected payoff conditional on observing $S = s$ is given by

$$B + \delta\mathbb{E}[X|S = s] - R + \sum_{i=1}^{I} \max\{P(G^{\varphi_i}) - \delta\mathbb{E}[\varphi_i(X)|S = s], 0\},$$

and her ex-ante payoff equals $B + \delta\mathbb{E}[X] - R + \sum_{i=1}^{I} V(G^{\varphi_i})$. Thus, the owner's problem boils down to maximization of information rents from trading all securities in $\boldsymbol{\varphi}$. For any sequence of securities $\boldsymbol{\varphi}$, the owner designs a signal S about X to solve the information design problem

$$\bar{V}(\boldsymbol{\varphi}) \equiv \sup_{\boldsymbol{G}^{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{\varphi}}} \sum_{i=1}^{I} V(G^{\varphi_i}). \tag{6}$$

The sequence of securities $\boldsymbol{\varphi}$ is optimal if it solves the security design problem

$$\sup_{\boldsymbol{\varphi} \in \boldsymbol{\Phi}} \bar{V}(\boldsymbol{\varphi}) \text{ s.t. } P(G^{\varphi_i}) \leq R_i \text{ for all } i = \overline{1, I}. \tag{7}$$

For our analysis, it is convenient to impose the constraints (5) in (7) but not in (6). Thus, the program (6) serves as an auxiliary program in that, for $\boldsymbol{\varphi}$ that violate these constraints, the value in (6) might not be attainable.

3 Preliminary Analysis: Escrow Problem

The main challenge in the analysis of problem (7) is that there is no analytically tractable characterization of the set of admissible CDFs $\boldsymbol{\mathcal{G}}^{\boldsymbol{\varphi}}$.[6] It is a standard result that each set $\mathcal{G}^{\varphi_i}$ is the set of all mean-preserving contractions of the prior distribution of security payoff, $H^{\varphi_i} \equiv H \circ \varphi_i^{-1}$. However, the owner is generally restricted by the fact that she designs a signal S about underlying cash flows, and so, a choice of a particular G^{φ_i} restricts her choices of $G^{\varphi_j}, j \neq i$. In other words, the inclusion (1) is generally strict.

To circumvent this challenge, we proceed as follows. We first solve a relaxed escrow problem that provides an upper bound on the value of the security design problem. In doing so, we show that the owner does not gain from splitting the payoffs of a single security into several securities, and hence, the solution to the escrow problem boils down to the case of $I = 1$, which is analytically tractable. We then show that common equity implements this upper bound in the security design problem, and so, it is optimal.

We start our analysis with the following benchmark. Suppose that, in order to continue operations, rather than raising R_i in each round, the owner only needs to raise R in total across all rounds $i = \overline{1, I}$. Formally, we consider the problem:

$$\sup_{\boldsymbol{\varphi} \in \boldsymbol{\Phi}, \boldsymbol{G}^{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{\varphi}}} \sum_{i=1}^{I} V\left(G^{\varphi_i}\right) \text{ s.t. } \sum_{i=1}^{I} P\left(G^{\varphi_i}\right) \leq R. \tag{8}$$

We call it the "escrow problem" and its interpretation is as follows. In period $t = 1$, the owner has access to a protected escrow and can put funds raised with security sales into it. These funds are then protected from diversion by the firm's management and are used to cover liquidity shocks. The owner can alternatively cover the liquidity shock herself and get repaid from the escrow in later rounds. In other words, the timing of liquidity shocks need not coincide with the timing of the funds raised through security sales (as in the baseline model).

This benchmark has a weaker form of the agency friction: the funds in the escrow are protected from the management diversion during stage $t = 1$, but if not fully spent by the end of it, they will be diverted by the insiders and do not accrue to the owner. Our main

[6] Johansen [1974] shows that the extreme rays in the cone of convex functions are dense in the cone, thereby precluding the existence of a simple characterization as in the single dimensional case.

result of the section is that the escrow problem has a simple solution: the owner sells *levered equity* and raises R all at once to fill up the escrow and then gradually draws on it as the liquidity shocks arrive.

3.1 Single Shock

We start the analysis of (8) with the case of a single shock, $I = 1$. To simplify the notation, we drop index i in this subsection. When $I = 1$, the owner designs a single security φ and the information design problem (6) becomes

$$\bar{V}(\varphi) \equiv \max_{G^\varphi \in \mathcal{G}^\varphi} V(G^\varphi). \tag{9}$$

The program (8) becomes

$$\sup_{\varphi\in\Phi, G^\varphi\in\mathcal{G}^\varphi} V(G^\varphi) \text{ s.t. } P(G^\varphi) \le R. \tag{10}$$

Here, for clarity, we denote by Φ the set $\mathbf{\Phi}$ in the case $I = 1$.

With a bit of an abuse of terminology, we refer to G^φ as a *signal distribution for security* φ (recall that G^φ is the CDF of $\mathbb{E}[\varphi_i(X)|S]$ so the signal interpretation is natural), and call G^φ solving (9) an *optimal signal distribution for* φ. The next lemma follows from Proposition 1 and Lemma 1 in Inostroza and Tsoy (2022) and characterizes the solutions to (9). We denote by $\mu^\varphi \equiv \mathbb{E}[\varphi(X)]$.

Lemma 1. *For any φ, let u^φ be the solution to*

$$\max_{u\le\varphi(\overline{x})} \{u : \mathcal{L}(y|\varphi,u) \ge 0, y \in [0,u]\} \tag{11}$$

$$\text{where } \mathcal{L}(y|\varphi,u) \equiv \mu^\varphi - \delta u - (1-\delta)\, y^{1/(1-\delta)} u^{-\delta/(1-\delta)} + \int_{-\infty}^{y} H^\varphi(f)\,\mathrm{d}f. \tag{12}$$

Then, $\boldsymbol{V}(\varphi) = \delta(u^\varphi - \mu^\varphi)$ and $G^\varphi \in \mathcal{G}^\varphi$ solves (9) if and only if (i) u^φ is the highest signal in the support of G^φ; (ii) trade occurs with probability one under G^φ (i.e., $P(G^\varphi) = \delta u^\varphi$). Further, an optimal signal always exists.

Lemma 1 characterizes the solutions to the information design program (9) for a single security φ. It allows us to write the constraint $G^\varphi \in \mathcal{G}^\varphi$ in (10) more explicitly using the $\mathcal{L}$ function in (12).

The lemma says that all optimal signal distributions share two economic properties. First, all such distributions share the same maximal element of their support, u^φ. That is, the owner commits not to learn realizations of the value of the security above u^{φ}. Second, any optimal signal distribution induces the liquidity supplier to offer the price $P\left(G^{\varphi}\right)=\delta u^{\varphi}$, which is accepted by all owner types.

Importantly, these two properties are not just necessary but are also sufficient to reach optimality. Indeed, any signal distribution satisfying the last two properties is optimal. Practically, this means that the commitment to some optimal signal might not be too demanding, and in many situations such a commitment might already be in place due to considerations other than liquidity needs. For example, corporations have accounting and risk management systems in place that commit them to learn information about risks. With limited resources, a corporation with these systems in place tends learns more granular information about the downside and more noisy information about the upside potential. We provide a detailed discussion of the realism of optimal signal distribution in our companion paper, Inostroza and Tsoy (2022).

Building on Theorem 1 in Inostroza and Tsoy (2022), we show next that levered equity is optimal in (10). To prove this, we introduce a notion of informational sensitivity. For securities φ and $\tilde{\varphi}$ with $\mu^{\tilde{\varphi}}=\mu^{\varphi}$, we say that security $\tilde{\varphi}$ is *more informationally sensitive* than φ if there exists $x^{*}\in[\underline{x},\overline{x}]$ such that $\tilde{\varphi}(x)\leq\varphi(x)$ for $x\in(\underline{x},x^{*})$ and $\tilde{\varphi}(x)\geq\varphi(x)$ for $x\in(x^{*},\overline{x})$. In words, $\tilde{\varphi}$ crosses φ once from below. We say that φ is a *levered equity* if $\varphi(X)=\max\{X-D,0\}$, for some $D>0$.

Intuitively, for any two securities φ and $\tilde{\varphi}$ with $\mu^{\tilde{\varphi}}=\mu^{\varphi}$, if $\tilde{\varphi}$ is more informationally sensitive than φ, then the distribution of security payoffs under $\tilde{\varphi}$, $H^{\tilde{\varphi}}$, consists of a mean preserving spread of the distribution of security payoffs under φ, H^{φ}. From the characterization of the information design problem in Lemma 1, we observe that increasing the informational sensitivity of the security, while keeping its expected payoff constant, relaxes the constraints in the program (11). Roughly, increasing the security's informational sensitivity provides more room to do information design. Levered equity is special in that it is the most informationally sensitive among all securities in Φ with the same expected payoff.[7] This proves its optimality.

Proposition 1. *For any $R>0$, there is a levered equity $\varphi^{R}(X)\equiv\max\left\{X-D^{R},0\right\}$, $D^{R}\geq0$, solving the escrow program (10). The solution is unique if*

$$\mu^{R}-\delta u^{R}>0,\tag{13}$$

where u^{R} solves (9) for φ^{R} and $\mu^{R}\equiv\mu^{\varphi^{R}}$. Further, if $\varphi^{R}(X)\neq X$ with positive probability

[7] This result is where we use that both $\varphi(x)$ and $x-\varphi(x)$ are increasing in x. See Inostroza and Tsoy (2022) for the analysis of securities that violate monotonicity of $x-\varphi(x)$.

and $u^R < \varphi^R(\overline{x})$, $u^R = R/\delta$.

A key insight of Inostroza and Tsoy (2022) is that, when the owner can design her private information, the informational sensitivity of securities is beneficial as it provides greater freedom in the choice of signals. This stands in contrast to the standard intuition from the security design literature with exogenous information, where least informationally sensitive securities tend to be optimal. To get intuition, consider the case of safe debt which is maximally informationally insensitive and imposes extreme restrictions from the information design perspective, as any signal about underlying cash flows gives no informational advantage to the owner. In turn, selling all cash flows to the liquidity supplier gives the owner great freedom in the design of information, as the variation in signal about X translates into the variation in the value of the security sold to the liquidity supplier. In the presence of the constraint $P(G^\varphi) \leq R$, selling all the cash flows might be suboptimal, because any amount raised in excess of R is lost to the agency friction. In this case, Proposition 1 establishes that it is optimal to issue levered equity.

3.2 General Escrow Problem

We next show that, if the owner could, she would prefer fill the escrow in a single round rather than spread security sales over several rounds. Formally,

Theorem 1. *For any* $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ *and* $\boldsymbol{G}^{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{\varphi}}$,

$$\sum_{i=1}^{I} V\left(G^{\varphi_i}\right) \leq \max_{G^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} V\left(G^{\hat{\varphi}}\right), \text{ where } \hat{\varphi}(X) \equiv \sum_{i=1}^{I} \varphi_i(X). \tag{14}$$

Further, selling a single security φ^R *in one of the rounds is optimal in the escrow problem* (*8*).

That is, for any collection of securities $\boldsymbol{\varphi}$, the owner weakly prefers selling a "bundle security" $\hat{\varphi}$ rather than selling securities in $\boldsymbol{\varphi}$ separately. This implies that the solution to (8) is creating an escrow by selling security φ^R in a single round, and then draw down this escrow as shocks arrive. Thus, the agency friction in (7) is potentially costly for the owner, as it forces her to raise funds separately for each liquidity shock.

Theorem 1 is in stark contrast to the optimality of tranching obtained in static models with exogenous private information (DeMarzo 2005, Biais and Mariotti 2005, Asriyan and Vanasco Forthcoming). Endogeneity of private information is key for this difference. To see this, suppose the owner perfectly learns X as in Biais and Mariotti (2005). Then, splitting security $\hat{\varphi}$ into a senior debt tranche $\varphi_1(X) = \min\{\hat{\varphi}(X), D\}$ and a junior equity tranche $\varphi_2(X) = \max\{\hat{\varphi}(X) - D, 0\}$ (for a certain D) and selling those tranches to different liquidity suppliers generally dominates selling $\hat{\varphi}(X)$. In other words, when $G^{\varphi_1}, G^{\varphi_2}$, and $G^{\hat{\varphi}}$ are all induced by the signal $S = X$, we have a reverse inequality to (14):

$$V(G^{\varphi_1}) + V(G^{\varphi_2}) \geq V\left(G^{\hat{\varphi}}\right). \tag{15}$$

The reason for this is that, when the owner knows X, $\hat{\varphi}$ might suffer illiquidity, namely, for high realizations of X, the owner rejects the liquidity supplier's offer $\hat{p}$ and there is no trade. In comparison, a less informationally sensitive debt $\varphi_1(X) = \min\{\hat{\varphi}(X), \hat{p}/\delta\}$ is always sold at price $\hat{p}$, and so, $V(G^{\varphi_1}) \geq V\left(G^{\hat{\varphi}}\right)$. In addition, the owner can extract some information rents from selling $\varphi_2(X) = \max\{\hat{\varphi}(X) - \hat{p}/\delta, 0\}$, which gives us (15).[8]

The idea of the proof is as follows. For a given signal S about X, the issuer faces a trade-off. On the one hand, offering separate securities gives more flexibility to the liquidity suppliers, as they can offer different prices (i.e., corresponding to different signals) for different φ_is which hurts the issuer. In other words, offering separate securities may reduce the issuer's ability to extract information rents.[9] On the other hand, selling securities separately can improve efficiency by increasing the gains from trade (as explained in the paragraph above), which can overweight the higher profits obtained by liquidity suppliers. The proof leverages the fact that when the owner has the flexibility to design information, she can guarantee the trade of any given security thereby removing the benefits of selling them separately. Thus, with endogenous information, the owner benefits from pooling multiple securities together and sell them as a single security while guaranteeing its sale and hence maximizing efficiency.

4 Optimality of Common Equity

4.1 Main Result

In this section, we present our central result about the optimality of common equity. We start with the following special property of linear securities.

[8]To the best of our knowledge, it is an open questions which securities are optimal in response to several liquidity shocks when the owner's information is exogenous. Note that, in the argument for inequality (15), we suppose that the owner can sell securities φ_1 and φ_2 to different liquidity suppliers, who are insulated and do not learn each others' trades. The analysis of optimal design would be further complicated if trades are observable. In contrast, as we show below, with optimal information design, the problem has a realistic solution, which is robust to whether trades are observable or not.

[9]Monotonicity assumption on φ_i's is important for Theorem 1. As a counter example, suppose $\varphi_1(X) + \varphi_2(X) = \hat{\varphi}(X)$ is a constant. The owner gets no information rents when selling $\hat{\varphi}$, but positive information rents when selling φ_1 and φ_2 separately.

Proposition 2. *Consider any* $\varphi \in \Phi$ *and any signal distribution* G^{φ} *solving program (9) for* φ*. Then, for any security of the form* $\tilde{\varphi}(X) = \alpha\varphi(X)$*,* $\alpha > 0$*, the signal distribution for* $\tilde{\varphi}$ *defined by* $\tilde{G}(z) = G^{\varphi}(z/\alpha)$ *is optimal for* $\tilde{\varphi}$*, and* $u^{\tilde{\varphi}} = \alpha u^{\varphi}$*. In particular, for* $\sum_{i=1}^{I} \alpha_i = 1, \alpha_i \geq 0, i = \overline{1, I}$*, and* $\varphi_i(X) = \alpha_i \varphi(X)$*,*

$$\sum_{i=1}^{I} \max_{G^{\varphi_i} \in \mathcal{G}^{\varphi_i}} V(G^{\varphi_i}) = \max_{G^{\varphi} \in \mathcal{G}^{\varphi}} V(G^{\varphi}).$$

Intuitively, for qualitative properties of optimal signal designs, it does not matter whether the security value is expressed in dollars, cents, thousands, or millions. The optimal signal simply scales accordingly. More formally, optimal signals are invariant with respect to linear transformations of the security under consideration. In turn, this invariance implies that selling a fraction or a multiple of a given security does not change the optimal signal. The same signal is optimal for the owner irrespective of whether she sells the whole security, a fraction, or a multiple of it.

We now state and prove our main result. Let us define the common equity as a share of levered equity. Formally, a *common equity with stake* α has a payoff of $\alpha \max\{X - D, 0\}$ for some $\alpha \in [0, 1]$ and $D \geq 0$. This definition captures two characteristic properties of the common equity: (i) it is junior to other payments (in this case, the payment of D); (ii) it is a share of cash flows after senior payments are made in full. Recall that φ^R denotes the levered equity that is optimal for raising a fixed amount R (see Proposition 1).

Theorem 2. *Consider any sequence of shocks* $\boldsymbol{R} = (R_i)_{i=1}^{I}$ *and any signal distribution* G^R *solving program (9) for* φ^R*, where* $R = \sum_{i=1}^{I} R_i$*. Let* $\boldsymbol{\varphi}^{\boldsymbol{R}} = \left(\varphi_i^{\boldsymbol{R}}\right)_{i=1}^{I}$ *be the collection of common equity securities with* $\varphi_i^{\boldsymbol{R}}(X) \equiv (R_i/R)\,\varphi^R(X), i = \overline{1, I}$*, and* $\boldsymbol{G}^{\boldsymbol{R}} \equiv \left(G_i^{\boldsymbol{R}}\right)_{i=1}^{I}$ *be given by* $G_i^{\boldsymbol{R}}(z) = G^R(Rz/R_i), i = \overline{1, I}$*. Then,* $\left(\boldsymbol{\varphi}^{\boldsymbol{R}}, \boldsymbol{G}^{\boldsymbol{R}}\right)$ *solves the problem (7).*

It is optimal for the owner to issue common shares as shocks arrive. The owner responds to each liquidity shock i by issuing additional common shares.[10]

The proof of Theorem 2 combines the solution to the escrow problem and the invariance property of linear securities. By Theorem 1, the escrow problem provides an upper bound on the owner's value in (7). That is, the owner cannot do better than by selling the levered equity security $\varphi^R(X) = \max\{X - R, 0\}$ in the first trading round and creating an escrow for all future shocks. The levered equity is optimal in this situation, because it allows for maximal freedom in information design. Yet, the escrow solution is not feasible as it violates

[10]Specifically, if n_0 is the number of shares outstanding at $t = 0$, then the owner issues n_i common shares in round i so that $R_i/R = n_i/\left(\sum_{i=0}^{I} n_i\right)$.

the constraints (5). In other words, in the presence of agency frictions, excess cash reserves are diverted by the management for private benefits.

Theorem 1 also reveals that deviating from the escrow solution is potentially costly. Selling a security in parts is costly for the owner, as it can hurt liquidity or information rents or both. An important concern the owner faces when selling multiple securities is that by learning information about the payoff of one of the securities she might compromise her position when trading the rest of the securities.

Proposition 2 reveals that selling shares of the security is immune to this problem. In particular, selling levered equity gradually (in the form of common shares) as shocks arrive allows the owner to implement the escrow solution in the sequential manner without violating the constraints (5). The owner simply chooses a signal S that is optimal for selling the whole levered equity allocated to cover shocks, φ^R, and the invariance property guarantees that the same signal is also optimal for selling any share of the levered equity to cover each shock R_i. This logic explains the optimality of equity ATMs.

4.2 Equity ATMs

Theorem 2 justifies the "equity ATMs" phenomenon. It predicts that in environments with asymmetric information, when the firm faces several liquidity shocks, it prefers to cover them by issuing common shares.

This prediction is in contrast to the classical pecking order theory that states that in the world of information asymmetry, when the firm needs external financial, debt is preferred to equity. Frank and Goyal (2003), Fama and French (2002) show that the pecking order theory fails to explain the financing of small high-growth firms who rely more on equity financing. This is surprising given that these firms are arguable most likely to have private information about their future prospects, and the pecking order theory should be most applicable to them.

Our theory resolves this contradiction. Equity ATMs are optimal when the liquidity needs are repeated, which is often true of small high-growth firms that do not generate cash flows and need cash injections to keep afloat. Further, as Lemma 1 suggests, the use of equity as ATMs is more likely in situations, where the firm has a more precise information about the downside risks rather than the upside potential. We expect this to be the case for firms that face material risk of financial distress, when survival becomes a top priority. This is again a situation in which small high-growth firms often find themselves in.

The use of equity as ATMs is documented empirically. One example is PIPE offerings that along with SEOs constitute a major source of secondary equity financing. Brophy et al. (2009) report that hedge funds actively participate in PIPE offerings along with other institutional investors. Such investors often buy shares at a substantial discount relative to the current share price, pointing to their significant market power. Floros and Sapp (2012) document that over 1995-2008, 71% of PIPE transactions are multiple issues with hedge funds becoming more dominant among repeated issuers. Multiple PIPE issuers tend to be levered, loss-generating small R&D intensive firms. The proceeds from issuances are used to cover financing deficits suggesting urgent liquidity needs. To sum up, PIPE offering illustrate the use of equity as ATMs and fit particularly well our assumptions.[11]

Another example concerns repeated SEOs. DeAngelo et al. (2010) argue that the classical pecking-order, trade-off, and market timing theories cannot explain the timing of SEOs. Rather, their evidence suggests that firms conduct SEOs to meet a near-term need for cash, akin to our equity as ATMs phenomenon. Typically SEOs are intermediated by the underwriter who often participates in the deal and creates a syndicate of institutional investors interested in the offering. Thus, the SEO application is also in line with an alternative interpretation of our model as raising financing from several liquidity constraint investors (see Remark 1).[12]

The final piece of evidence in support of equity ATMs comes from DeAngelo et al. (2011). They structurally estimate a dynamic capital structure model, which incorporates major financing frictions, including tax benefits and default costs of debt as well as costs of equity financing due to adverse selection. DeAngelo et al. (2011) report that the model matches remarkably well all the targeted moments in the data, except for the average equity issuance, where the simulated value differs significantly from the data. They suggest that the model fit can be improved if equity bears an additional benefit, that increases the share of firms issuing equity predicted by the model. Our theory provides such a novel benefit of equity financing.

4.3 Suboptimality of Tranching

The classical literature on security design suggests that a natural candidate to address the issuer's liquidity problems in environments like ours with asymmetric information is the collection of *debt tranch*es. In practice, the originator sells the cash flows rights of a pool of assets to a special purpose vehicle (SPV) which, in turn, sells securities with different seniorities, the so-called tranches, to the market. The originator typically keeps the most

[11]Interestingly, PIPE investors request board seats in only 15.3% of first-time offerings, which goes down to 6.5% towards the sixth offering. This suggests that adverse selection rather than moral hazard is the key friction in such transactions.

[12]Further, as we argue in the next section, in situations when the issuer raises a fixed amount, the incentives of the underwriter are tilted towards those of the members of the syndicate.

junior tranche and the SPV then sell the debt tranches to the investors (Gorton and Metrick (2013)). DeMarzo (2005) argue that, under some conditions, pooling and tranching can help the issuer create low-risk and highly liquid securities. Below, we provide conditions under which common equity strictly dominates debt tranching for problem (7). These results reveal a novel mechanism of how common equity dominates other securities.

Consider an arbitrary collection of securities $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^{I} \in \boldsymbol{\Phi}$. We say that $\boldsymbol{\varphi}$ is a *collection of debt tranches of* X if there is a non-negative and weakly increasing sequence $(D_j)_{j=1}^{I}$, and a permutation $\gamma(\cdot)$ of $\{1,\ldots,I\}$ such that, for all $j = \overline{1,I}$, $\sum_{i=1}^{j} \varphi_{\gamma(i)}(X) = \min\{X, D_j\}$.

In what follows, we suppose that $\varphi^R(X) \neq X$ with positive probability and $u^R < \varphi^R(\overline{x})$. The former condition means that not all cash flows are exhausted to cover the liquidity shocks. The latter condition means that the adverse selection is sufficiently severe that, even with the information design, the owner has to sell securities at a discount to their maximal value $\delta\varphi^R(\overline{x})$. Further, we suppose that $\mu^R - \delta u^R > 0$. Proposition 1 states that under these conditions, the solution to the escrow problem with $I = 1$ is unique and given by φ^R, and further, the security offering is sufficient to cover the whole liquidity shock.

Proposition 3. *Consider any sequence of shocks* $\boldsymbol{R} = (R_i)_{i=1}^{I}$ *such that* $\mu^R - \delta u^R > 0$. *Any collection of debt tranches of* X*,* $\boldsymbol{\varphi}$*, such that* $\mathbb{P}\left[\sum_{i=1}^{I} \varphi_i(X) \neq X\right] > 0$*, is strictly dominated by* $\boldsymbol{\varphi}^{\boldsymbol{R}}$*.*

4.4 Strict Optimality

We show next that, in many circumstances, common equity is the unique solution to the problem (7). We start with an observation that follows directly from Theorem 2, and that helps us simplify the analysis.

Corollary 1. *Consider a sequence of shocks* $\boldsymbol{R} = (R_i)_{i=1}^{I}$ *and suppose that* $\mu^R - \delta u^R > 0$. *Any collection of securities* $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ *solving program (7) must satisfy* $\sum_{i=1}^{I} \varphi_i(X) = \varphi^R(X)$*.*

Corollary 1 establishes that any solution $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^{I}$ to the owner's problem must add up to the levered equity security φ^R that solves the escrow problem.

Before stating the main result of this section, we provide an intermediate result. The next lemma provides an upper bound for the owner's payoff that follows from the inclusion (1).

Lemma 2. *For any* $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$*,* $\max\limits_{G^{\boldsymbol{\varphi}} \in \mathcal{G}^{\boldsymbol{\varphi}}} \sum_{i=1}^{I} V(G^{\varphi_i}) \leq \sum_{i=1}^{I} \max\limits_{G^{\varphi_i} \in \mathcal{G}^{\varphi_i}} V(G^{\varphi_i})$*.*

In words, the owner values the flexibility in choosing a different signal S_i for each $\varphi_i(X)$ rather than a single signal S for all of them. One of the major difficulties in solving problem (7) originates from the lack of a useful characterization of the possible distributions available to the owner, $\mathcal{G}^{\boldsymbol{\varphi}}$, for the case where $I \geq 2$. We bypass this issue by focusing instead in a relaxation of the owner's problem where she can choose a different signal S_i for each $\varphi_i(X)$ rather than a single signal S about X for all of them. The relaxed problem thus consists in a sequence of independent problems whose solution is characterized in Lemma 1.

To get some traction, we restrict attention to the case with two cash flow realizations, $\mathcal{X} = \{\underline{x}, \overline{x}\}$, and show that, in this case, common equity is the unique solution to the problem (7).

Proposition 4. *Suppose $I \geq 2$ and $\mathcal{X} = \{\underline{x}, \overline{x}\}$. For any $\boldsymbol{R} = (R_i)_{i=1}^{I}$ so that $R < \delta\varphi^R(\overline{x})$, $\boldsymbol{\varphi}^{\boldsymbol{R}}$ in Theorem 2 is the unique collection of securities solving (7).*

Proposition 4 builds on a strong result. When $\mathcal{X} = \{\underline{x}, \overline{x}\}$ and $R < \delta\varphi^R(\overline{x})$, any collection of securities $\boldsymbol{\varphi}$ that differs from $\boldsymbol{\varphi}^{\boldsymbol{R}}$ with positive probability satisfies $\sum_{i=1}^{I} \max_{G^{\varphi_i} \in \mathcal{G}^{\varphi_i}} V(G^{\varphi_i}) < \bar{V}\left(\boldsymbol{\varphi}^{\boldsymbol{R}}\right)$. That is, even if the owner had the flexibility of independently design the optimal signal before trading with each liquidity supplier, for any collection of securities different from the optimal common equity $\boldsymbol{\varphi}^{\boldsymbol{R}}$, her payoff would be strictly dominated by the one she can achieve by selling common equity and learning a single signal about X.

5 Robustness and Extensions

Random Liquidity Shocks. In the baseline model, we assume that the sequence of shocks $\boldsymbol{R}$ is common knowledge from the onset. We can relax this assumption as follows. Suppose $\boldsymbol{R}$ is distributed with the CDF $H_{\boldsymbol{R}}$ and is not known to any market participant at the design stage at $t = 0$. Denote by $R \equiv \sum_{i=1}^{I} R_i$ the total liquidity that the firm needs to raise at $t = 1$. We assume that, while the magnitude of each particular shock is uncertain, the aggregate liquidity need R is common knowledge at $t = 0$. Thus, there is uncertainty about the timing and the size of shocks, but not the aggregate liquidity need. $\boldsymbol{R}$ becomes publicly observed at the beginning of the trading stage, $t = 1$.

This assumption is in line with how shelf-registration is normally done, where firms specify the total dollar amount of securities that can be issued under shelf registration. This practice reflects the idea that, while the exact sequence of shocks might be uncertain at the time of shelf registration submission, the issuer often has a fairly good, andperhaps conservative, estimate of potential liquidity shortages that the company will face in the near future.

At $t = 0$, the owner commits to a state contingent issuance plan that maps each realization of shocks, $\boldsymbol{R}$, into securities $\boldsymbol{\varphi}(\boldsymbol{R}) \in \boldsymbol{\Phi}$ to cover them. Then, since Theorem 2 holds for any sequence $\boldsymbol{R}$ and φ^R depends only on the total liquidity need R, even when $\boldsymbol{R}$ is uncertain at $t = 0$, it is still optimal for the owner to choose the signal G^R and issue common equity in response to any realization $\boldsymbol{R}$ that slices the levered equity φ^R in proportion to the individual shock realizations, R_is, as described in Theorem 2. In other words, it is not necessary to know at $t = 0$ the exact realizations of R_is, as long as the aggregate R is known, and so, the owner can allocate φ^R to cover her liquidity needs and design her information appropriately for selling φ^R.

Key to this robustness is the invariance property of linear securities in Proposition 2. Selling any slice of φ^R optimally does not require a change of private information about X. This gives common equity flexibility in responding to uncertain shocks $\boldsymbol{R}$.

Alternative Trading Environments. In the baseline model, we maximally restrict the information that the liquidity suppliers learn about other security sales. Here, we relax this assumption. We modify the baseline model and assume that the securities are sold sequentially with *post-trade transparency.* That is, at any round $i \in \{2, ..., I\}$, the liquidity supplier i perfectly observes the securities that have been offered at all previous rounds $j < i$, together with the price offered by the liquidity supplier at that round and whether or not the owner accepted it. We argue that common equity is still optimal in this environment. To see this, fix a given collection of securities $\boldsymbol{\varphi} \in \Phi$, and consider an arbitrary signal S about X. Let $\boldsymbol{G}^{\boldsymbol{\varphi}} = (G^{\varphi_i})_{i=1}^{I}$ be the induced distributions of the value of the securities.

Suppose now that, at the beginning of each round i, liquidity supplier i observes a signal S_i about X, weakly less informative than signal S. In this new environment with sequential trading and post-trade transparency, signal S_i captures the information learned by liquidity supplier i from observing the outcome of previous rounds. For example, liquidity supplier i updates her beliefs upwards about the value of security $\varphi_i(X)$ after observing that the owner rejects a price $p_j = P(G^{\varphi_j})$ for security $\varphi_j(X)$ at round j. The arguments below, however, apply more generally for arbitrary collection of private signals $(S_i)_{i=1}^{I}$.

Following the same arguments used to establish the Theorem 1, one can prove that the owner's payoff from selling any collection of securities $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ and picking signals S and S_is about X is bounded from above by $\max_{G^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} V\left(G^{\hat{\varphi}}\right)$ where $\hat{\varphi}(X) \equiv \sum_{i=1}^{I} \varphi_i(X)$. This implies, in particular, that the owner's payoff under sequential trading and post-trade transparency is bounded from above by $\bar{V}\left(\varphi^R\right)$, by Proposition 1. Consider then the collection of common equity securities $\boldsymbol{\varphi}^{\boldsymbol{R}} = \left(\varphi_i^{\boldsymbol{R}}\right)_{i=1}^{I}$ in Theorem 2 and let $\boldsymbol{G}^{\boldsymbol{R}} \equiv \left(G_i^{\boldsymbol{R}}\right)_{i=1}^{I}$ and (one of) the associated optimal signal distribution of the value of the securities such that $\left(\boldsymbol{\varphi}^{\boldsymbol{R}}, \boldsymbol{G}^{\boldsymbol{R}}\right)$

solves the problem (7). Because, under $\boldsymbol{G^R}$, each security $\varphi_i^{\boldsymbol{R}}$ is traded with probability 1 in the absence of post-trade transparency, each liquidity supplier i can perfectly predict the price $P\left(G_i^{\boldsymbol{R}}\right)$ that takes place at each of the previous rounds even without observing the previous outcomes. Thus, the information learned from observing the previous securities and prices becomes redundant and does not lead to any bayesian updating. The result thus follows from the fact that, regardless of the round at which each liquidity supplier trades, when the owner chooses $\left(\boldsymbol{\varphi^R}, \boldsymbol{G^R}\right)$, their information set is the same.

Debt Issuance. We next study an extension which gives rise to debt being issued alongside common shares. This result provide predictions about circumstances in which debt or common equity is issued. Importantly and distinctly from the literature, such predictions are obtained without restricting the class of securities to solely debt and equity.

In the baseline model, common equity is junior to the debt with face value D retained by the owner. With a slight modification of the baseline model, we can imagine that this debt would be sold to outside investors (due to existence of gains from trade) in later stages. Specifically, suppose that there are two rounds (i.e., $I = 2$) and let $\boldsymbol{R} = (R_1, R_2)$. Round 1 represents the "growth stage," when the firm requires liquidity injections and suffers from various market frictions. In particular, and as in the baseline model, the liquidity supplier is monopolistic and holds all the bargaining power. Round 2, in turn, captures the "maturity stage." At this stage, the firm matures in that the securities can be sold in a competitive market.

In this modification, it is optimal for the owner to sell levered equity $\varphi_1(X) \equiv \max\left\{X - D^{R_1}, 0\right\}$ in round 1, and debt $\varphi_2(X) = \min\{X, d\}$ in round 2, with $D^{R_1}, d > 0$.

Proposition 5. *Suppose that $I = 2$ and that the liquidity supplier at round 2 is perfectly competitive, then, the owner optimally sells $\varphi_1(X) \equiv \varphi^{R_1}(X)$ and $\varphi_2(X) = \min\{X, d\}$, where $d \geq 0$ is such that $\mu^{\varphi_2} = R_2$.*

Intermediation through Underwriters. In the baseline model, liquidity suppliers have full bargaining power. This assumption is relevant in the context of PIPE offerings, where funding is raised from a handful of institutional investors, such as hedge funds and private equity funds, who wield significant market power (e.g., they negotiate significant discounts relative to the current share price, see Brophy et al. 2009).

SEOs are intermediated by underwriters who possess significant market power (illustrated by the significant discount of roughly 5% on the current stock price). In a typical offering, the lead underwriter negotiates the price with the issuer and markets the shares to a broad group of investors. The underwriter is motivated by both his direct fee (a percentage of the proceeding from issuance) and reputational incentives for giving a good deal to other investors. In this case, his expected payoff is

$$\underbrace{\alpha R_i}_{\text{fee}} + \lambda \underbrace{\pi\left(p|G^{\varphi_i}\right)}_{\text{reputational incentives}}.$$

DeAngelo et al. 2010 shows that SEOs are often a response to immediate cash needs. In these situation, it is reasonable to suppose that the company tries to raise just enough money to cover those needs (to avoid money being wasted by the management), and the expected underwriter's fee is fixed at the size of these cash needs. In these circumstances, the reputational incentives are the main driver in negotiation with the issuer – a lower price of the offering means that other investors participating in the SEO would get more shares for the same investment. Thus, the underwriter's object is well described by the monopolistic liquidity supplier's object in our model.

Note that if the liquidity suppliers are competitive in all rounds, then it is optimal for the owner to learn nothing about cash flows and there are many optimal security designs: any security with $R_i = \mu^{\varphi_i}$ is optimal. However, the competitive case, while theoretically interesting, is unrealistic in the context of small firms raising financing in response to the immediate and pressing cash needs. Further, it contradicts the significant discount at the current share price that arises in both PIPE offerings and SEOs. Overall, our assumption of significant market power of liquidity suppliers is more in line with reality.

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Appendix

We relegate the proofs of auxiliary lemmas to Online Appendix.

Lemma 3. *Consider any* φ_i *and any* $G^{\varphi_i} \in \mathcal{G}^{\varphi_i}$ *such that* $P(G^{\varphi_i}) > R_i$ *and* $V^{R_i}(G^{\varphi_i}) > 0$*. Let* $\tilde{\varphi}_i(X) \equiv \alpha\varphi_i(X)$ *with* $\alpha = R_i/P(G^{\varphi_i}) \in (0,1)$*, and* $\tilde{G}(\tilde{z}) \equiv G^{\varphi_i}(\tilde{z}/\alpha)$ *for all* $\tilde{z}$*. Then,* $\tilde{G} \in \mathcal{G}^{\tilde{\varphi}_i}$*,* $P\left(\tilde{G}\right) = R_i$ *and* $V\left(\tilde{G}\right) > V^{R_i}(G^{\varphi_i})$*. Further, both* G^{φ_i} *and* $\tilde{G}$ *are generated by the same signal* S *about* X*.*

Proof of Lemma 3. By Theorems 3.A.1 and 3.A.4 in Shaked and Shanthikumar (2007a), $\tilde{G} \in \mathcal{G}^{\tilde{\varphi}_i}$ is equivalent to $\int_{-\infty}^{y} \tilde{G}(z)\,\mathrm{d}z \le \int_{-\infty}^{y} H^{\tilde{\varphi}_i}(f)\,\mathrm{d}f$ for all y and $\mathbb{E}_{\tilde{G}}[Z] = \mathbb{E}_H[\tilde{\varphi}_i(X)]$. These two conditions are in turn equivalent to $\int_{-\infty}^{y} G^{\varphi_i}(z)\,\mathrm{d}z \le \int_{-\infty}^{y} H^{\varphi_i}(f)\,\mathrm{d}f$ for all y and $\mathbb{E}_{G^{\varphi}}[Z] = \mathbb{E}_H[\varphi_i(X)]$ (by construction of $\tilde{\varphi}_i$ and $\tilde{G}$). Thus, $G^{\varphi_i} \in \mathcal{G}^{\varphi_i}$ implies that $\tilde{G} \in \mathcal{G}^{\tilde{\varphi}_i}$. Moreover, (2) implies that $P\left(\tilde{G}\right) = \alpha P(G^{\varphi_i}) = R_i$. Using the change of variables $z = \tilde{z}/\alpha$,

$$
\begin{aligned}
V\left(\tilde{G}\right) &= \int_{-\infty}^{R_i/\delta} (R_i - \delta\tilde{z})\,\mathrm{d}\tilde{G}(\tilde{z}) \\
&= \int_{-\infty}^{\alpha P(G^{\varphi_i})/\delta} (R_i - \delta\tilde{z})\,\mathrm{d}G^{\varphi_i}(\tilde{z}/\alpha) \\
&= \int_{-\infty}^{P(G^{\varphi_i})/\delta} (R_i - \delta\alpha z)\,\mathrm{d}G^{\varphi_i}(z) \\
&= \int_{-\infty}^{R_i/\delta} (R_i - \delta\alpha z)\,\mathrm{d}G^{\varphi_i}(z) + \int_{R_i/\delta}^{P(G^{\varphi_i})/\delta} (R_i - \delta\alpha z)\,\mathrm{d}G^{\varphi_i}(z) \\
&> \int_{-\infty}^{R_i/\delta} (R_i - \delta z)\,\mathrm{d}G^{\varphi_i}(z) + \int_{R_i/\delta}^{P(G^{\varphi_i})/\delta} \underbrace{(R_i - \alpha P(G^{\varphi_i}))}_{=0}\,\mathrm{d}G^{\varphi_i}(z) \\
&= V(G^{\varphi_i}),
\end{aligned}
$$

where the inequality follows from $\alpha < 1$ and G^{φ_i} assigns positive probability to $z < P(G^{\varphi_i})/\delta$ (by $V(G^{\varphi_i}) > 0$). Finally, to see that both G^{φ_i} and $\tilde{G}$ are generated by the same signal, let S be the signal about X inducing G^{φ_i}. The last statement follows from $\tilde{G}(\tilde{z}) = \mathbb{P}[\mathbb{E}[\tilde{\varphi}_i(X)|S] \leq \tilde{z}] = G^{\varphi_i}(\tilde{z}/\alpha)$ for all $\tilde{z}$. □

Proof of Lemma 1. The proof follows from Proposition 1 and Lemma 1 in Inostroza and Tsoy (2022), which in turn builds on Kartik and Zhong (2023). □

The following lemma is a strict version of Lemma 2 in Inostroza and Tsoy (2022).

Lemma 4. *Consider any $\varphi \in \Phi$ such that $\varphi(X) \neq X$ with positive probability and $u^{\varphi} < \varphi(\overline{x})$. If $\Delta > 0$ is such that $\tilde{\varphi}(X) = \varphi(X) + \Delta \in \Phi$, then $\bar{V}(\varphi) < \bar{V}(\tilde{\varphi})$ and $u^{\varphi} + \Delta < u^{\tilde{\varphi}}$. Further, $u^{\tilde{\varphi}}$ changes continuously with Δ.*

Proof. We will show that

$$\mathcal{L}(y|\tilde{\varphi}, u^{\varphi} + \Delta) > 0, y \in [0, u^{\varphi} + \Delta]. \tag{16}$$

Then, by Lemma 1, there is $\varepsilon > 0$ such that $u^{\tilde{\varphi}} > u^{\varphi} + \Delta + \varepsilon$, hence, $\boldsymbol{V}(\tilde{\varphi}) = \delta\left(u^{\tilde{\varphi}} - \mu^{\tilde{\varphi}}\right) > \delta(u^{\varphi} + \Delta + \varepsilon - (\mu^{\varphi} + \Delta)) > \delta(u^{\varphi} - \mu^{\varphi}) = \boldsymbol{V}(\varphi)$.

Let us prove (16). For $y \in [0, \varphi(\underline{x}) + \Delta)$, $\mathcal{L}(y|\tilde{\varphi}, u^{\varphi} + \Delta) = \mu^{\varphi+\Delta} - \delta(u^{\varphi} + \Delta) - (1-\delta)y^{1/(1-\delta)}(u^{\varphi} + \Delta)^{-\delta/(1-\delta}$ is decreasing in y, hence, it is sufficient to verify that $\mathcal{L}(y|\tilde{\varphi}, u^{\varphi} + \Delta) > 0, y \in [\varphi(\underline{x}) + \Delta, u^{\varphi} + \Delta]$, or equivalently, $\mathcal{L}(y + \Delta|\tilde{\varphi}, u^{\varphi} + \Delta) > 0, y \in [\varphi(\underline{x}), u^{\varphi}]$. More explicitly,

$$\mu^{\varphi} + \Delta - \delta(u^{\varphi} + \Delta) - (1-\delta)(y + \Delta)^{1/(1-\delta)}(u^{\varphi} + \Delta)^{-\delta/(1-\delta)} + \int_{\varphi(\underline{x})+\Delta}^{y+\Delta} H^{\tilde{\varphi}}(f)\,\mathrm{d}f > 0, y \in [\varphi(\underline{x}), u^{\varphi}].$$

which is equivalent to

$$\mathcal{L}(y|\varphi, u^{\varphi}) + \underbrace{\Delta(1-\delta) + (1-\delta)y^{1/(1-\delta)}(u^{\varphi})^{-\delta/(1-\delta)} - (1-\delta)(y + \Delta)^{1/(1-\delta)}(u^{\varphi} + \Delta)^{-\delta/(1-\delta)}}_{\equiv \psi(y,\Delta)} > 0, y \in [\varphi(\underline{x}), u^{\varphi}].$$

This inequality holds for $y = u^{\varphi}$ by $\psi(u^{\varphi}, \Delta) = 0$ and

$$\begin{aligned}\mathcal{L}(u^{\varphi}|\varphi, u^{\varphi}) &= \mu^{\varphi} - u^{\varphi} + \int_{\varphi(\underline{x})}^{u^{\varphi}} H^{\varphi}(f)\,\mathrm{d}f \\ &= \mu^{\varphi} - u^{\varphi}(1 - H^{\varphi}(u^{\varphi})) - \int_{\varphi(\underline{x})}^{u^{\varphi}} f\,\mathrm{d}H^{\varphi}(f) \\ &= \mathbb{E}[\varphi(X)] - \mathbb{E}[\min\{\varphi(X), u^{\varphi}\}] > 0.\end{aligned}$$

Since $\psi(y, 0) = 0$ and $\mathcal{L}(y|\varphi, u^{\varphi}) \geq 0$ for $y \in [\varphi(\underline{x}), u^{\varphi})$, to show that the inequality also holds for $y \in [\varphi(\underline{x}), u^{\varphi})$, it is sufficient to show that $\psi_{\Delta}(y, \Delta) > 0$ for all such ys and $\Delta > 0$. We have

$$\psi_{\Delta}(y, \Delta) = 1 - \delta - \left(\frac{y + \Delta}{u^{\varphi} + \Delta}\right)^{\delta/(1-\delta)} + \delta\left(\frac{y + \Delta}{u^{\varphi} + \Delta}\right)^{1/(1-\delta)},$$

which by straightforward differentiation is strictly decreasing in y, and so, indeed, $\psi_\Delta(y,\Delta) > \psi_\Delta(u^\varphi,\Delta) = 0$.

Finally, let us show that $u^{\tilde{\varphi}}$ cannot discontinuously jump from u^φ as Δ increases (here, the dependence of $\tilde{\varphi}$ on Δ is implicit in the notation). Suppose to contradiction that there is $d > 0$ such that $u^\varphi + d < u^{\tilde{\varphi}}$ for all $\Delta > 0$. Let $\hat{y} \in [0, u^\varphi]$ be such that $\mathcal{L}(\hat{y}|\varphi, u^\varphi) = 0$. By the argument above $\mathcal{L}(u^\varphi|\varphi, u^\varphi) > 0$ and so, $\hat{y} < u^\varphi$. Then,

$$\begin{aligned}
\mathcal{L}\left(\hat{y}+\Delta\middle|\tilde{\varphi},u^{\tilde{\varphi}}\right) =& \mu^{\tilde{\varphi}} - \delta u^{\tilde{\varphi}} - (1-\delta)(\hat{y}+\Delta)^{1/(1-\delta)}\left(u^{\tilde{\varphi}}\right)^{-\delta/(1-\delta)} + \int_{-\infty}^{\hat{y}+\Delta} H^{\tilde{\varphi}}(f)\,\mathrm{d}f \\
=& \mu^{\tilde{\varphi}} - \delta u^{\tilde{\varphi}} - (1-\delta)(\hat{y}+\Delta)^{1/(1-\delta)}\left(u^{\tilde{\varphi}}\right)^{-\delta/(1-\delta)} + \int_{-\infty}^{\hat{y}} H^{\varphi}(f)\,\mathrm{d}f \\
=& \mu^{\tilde{\varphi}} - \delta u^{\varphi} - (1-\delta)(\hat{y}+\Delta)^{1/(1-\delta)}\left(u^{\varphi}\right)^{-\delta/(1-\delta)} \\
& - \int_{u^\varphi}^{u^{\tilde{\varphi}}} \left(\delta - \delta\left(\frac{\hat{y}+\Delta}{u}\right)^{1/(1-\delta)}\right)\mathrm{d}u + \int_{-\infty}^{\hat{y}} H^{\varphi}(f)\,\mathrm{d}f \\
<& \mu^{\tilde{\varphi}} - \delta u^{\varphi} - (1-\delta)(\hat{y}+\Delta)^{1/(1-\delta)}\left(u^{\varphi}\right)^{-\delta/(1-\delta)} \\
& - d\delta\left(1 - \left(\frac{\hat{y}+\Delta}{u^\varphi}\right)^{1/(1-\delta)}\right) + \int_{-\infty}^{\hat{y}} H^{\varphi}(f)\,\mathrm{d}f.
\end{aligned}$$

Taking the limit $\Delta \to 0$,

$$\mathcal{L}\left(\hat{y}\middle|\tilde{\varphi},u^{\tilde{\varphi}}\right) < \mathcal{L}(\hat{y}|\varphi,u^\varphi) - d\delta\left(1-\left(\frac{\hat{y}}{u^\varphi}\right)^{1/(1-\delta)}\right) = -d\delta\left(1-\left(\frac{\hat{y}}{u^\varphi}\right)^{1/(1-\delta)}\right) < 0,$$

which contradicts that $u^{\tilde{\varphi}}$ solves (9) for $\tilde{\varphi}$. Therefore, $u^{\tilde{\varphi}}$ is continuous in Δ. □

Proof of Proposition 1. 1) Consider φ solving (10). Since $\varphi \in \Phi$, there is $\hat{\varphi}(X) = \max\{X - D, 0\}$, $D \geq 0$, such that $\mu^{\hat{\varphi}} = \mu^\varphi$. Since $\hat{\varphi}$ is more informationally sensitive than φ, by Theorem 1 in Inostroza and Tsoy (2022), $\mathcal{G}^{\hat{\varphi}} \supseteq \mathcal{G}^\varphi$ and $\bar{V}(\hat{\varphi}) \geq \bar{V}(\varphi)$. Let $G^{\hat{\varphi}}$ be an optimal signal distribution for $\hat{\varphi}$. If $P\left(G^{\hat{\varphi}}\right) \leq R$, then $\hat{\varphi}$ is the desired security. If instead $P\left(G^{\hat{\varphi}}\right) > R$, let $\alpha \equiv R/P\left(G^{\hat{\varphi}}\right) < 1$ and define $\varphi_\alpha(X) \equiv \alpha\hat{\varphi}(X)$. By Lemma 3, $G_\alpha(\tilde{z}) \equiv G^{\hat{\varphi}}(\tilde{z}/\alpha)$ is feasible for φ_α and $P(G_\alpha) = \alpha P\left(G^{\hat{\varphi}}\right) = R$. Moreover, $\bar{V}(\varphi_\alpha) = V(G_\alpha) > V^R\left(G^{\hat{\varphi}}\right) = \bar{V}(\hat{\varphi}) \geq \bar{V}(\varphi)$. Finally, let $\tilde{\varphi}(X) \equiv \max\left\{X - \tilde{D}, 0\right\}, \tilde{D} \geq 0$, such that $\mu^{\tilde{\varphi}} = \mu^{\varphi_\alpha}$. Because $\alpha < 1$, $\tilde{\varphi}$ is more informationally sensitive than φ_α, and therefore, by Theorem 1 in Inostroza and Tsoy (2022), $\mathcal{G}^{\tilde{\varphi}} \supseteq \mathcal{G}^{\varphi_\alpha}$ and $\bar{V}(\tilde{\varphi}) \geq \bar{V}(\varphi_\alpha) > \bar{V}(\varphi)$ thereby establishing the first statement.

2) Consider any $\varphi \in \Phi$ and $\hat{\varphi}(X) = \max\{X - D, 0\}$, $D \geq 0$, such that $\mu^{\tilde{\varphi}} = \mu^\varphi$. Suppose that $\varphi(X) \neq \hat{\varphi}(X)$ with positive probability. Since $\varphi \in \Phi$, φ is differentiable almost everywhere and $\varphi'(x) \leq \hat{\varphi}'(x) = 1$ for almost all $x \in [\max\{\underline{x}, D\}, \overline{x}]$. Hence, if it were $\varphi(\max\{\underline{x}, D\}) \leq \hat{\varphi}(\max\{\underline{x}, D\})$, then $\varphi(x) \leq \hat{\varphi}(x)$ for $x \in [\max\{\underline{x}, D\}, \overline{x}]$, and so, either $\varphi(X) = \hat{\varphi}(X)$ with probability one or $\mu^\varphi < \mu^{\hat{\varphi}}$, which contradicts our construction of $\hat{\varphi}$. Thus, $\varphi(\max\{\underline{x}, D\}) > \hat{\varphi}(\max\{\underline{x}, D\})$. Similarly, if it were $\varphi(\overline{x}) \geq \hat{\varphi}(\overline{x})$, then $\varphi(x) \geq \hat{\varphi}(x)$ for $x \in [\underline{x}, \overline{x}]$, and so, either

$\varphi(X) = \hat{\varphi}(X)$ with probability one or $\mu^{\varphi} > \mu^{\hat{\varphi}}$, which again contradicts our construction of $\hat{\varphi}$. Thus, $\hat{\varphi}(\overline{x}) > \varphi(\overline{x})$. Therefore, for $\varepsilon = \min\{\varphi(\max\{\underline{x}, D\}) - \hat{\varphi}(\max\{\underline{x}, D\}), \varphi(\overline{x}) - \hat{\varphi}(\overline{x})\} > 0$, $H^{\hat{\varphi}}(f) > H^{\varphi}(f)$ for $f \in (\hat{\varphi}(\underline{x}), \hat{\varphi}(\underline{x}) + \varepsilon)$ and $H^{\hat{\varphi}}(f) < H^{\varphi}(f) = 1$ for $f \in (\hat{\varphi}(\overline{x}) - \varepsilon, \hat{\varphi}(\overline{x}))$. Theorem 1 in Inostroza and Tsoy (2022) then implies that

$$\int_{-\infty}^{y} H^{\hat{\varphi}}(f)\,\mathrm{d}f > \int_{-\infty}^{y} H^{\varphi}(f)\,\mathrm{d}f \text{ for all } y \in (\hat{\varphi}(\underline{x}), \hat{\varphi}(\overline{x})). \tag{17}$$

Thus, $\mathcal{L}(y|\hat{\varphi}, u^{\varphi}) > 0$ for all $y \in (0, u^{\varphi})$. Further, note that $0 \leq \pi(\delta u^{\varphi}|G^{\varphi}) = \mu^{\varphi} - \delta u^{\varphi}$. Unless $\mu^{\varphi} = \delta u^{\varphi}$ (in which case $\mathcal{L}(0|\varphi, u^{\varphi}) = \mathcal{L}(0|\hat{\varphi}, u^{\varphi}) = 0$), $u^{\hat{\varphi}} > u^{\varphi}$ and $\bar{V}(\hat{\varphi}) = \delta\left(u^{\hat{\varphi}} - \mu^{\varphi}\right) > \delta(u^{\varphi} - \mu^{\varphi}) = \bar{V}(\varphi)$. This implies that φ^R is the unique solution to (10) whenever $\mu^R > \delta u^R$.

3) Suppose $\varphi^R(X) \neq X$ with positive probability (which implies $D^R > 0$) and $u^R < \varphi^R(\overline{x})$. Suppose to contradiction $u^R < R/\delta$. By Lemma 4, there is $\Delta > 0$ such that, for $\tilde{\varphi}(X) \equiv \varphi^R(X) + \Delta$, $\bar{V}\left(\varphi^R\right) < \bar{V}(\tilde{\varphi})$ and for sufficiently small Δ, $u^{\tilde{\varphi}} \leq R/\delta$, which contradicts the optimality of φ^R. Therefore, it must be $u^R = R/\delta$. □

Proof of Theorem 1. Let S be a signal about X generating $\boldsymbol{G^{\varphi}}$ and $G^{\hat{\varphi}}$ be the induced distribution of $\mathbb{E}[\hat{\varphi}(X)|S]$. By Theorem 3.A.4 in Shaked and Shanthikumar (2007b), we can choose S such that (i) $\mathcal{S} = \mathcal{X}$ and $S = \mathbb{E}[X|S]$; (ii) the conditional distribution $[X|S = s]$ first-order stochastically dominates $[X|S = t]$ for all $s > t$.

Consider first an auxiliary problem where the owner offers the bundled security $\hat{\varphi}$ to a single liquidity supplier. In the Online Appendix, we prove the following claim.

Claim 1. $\pi\left(p|G^{\hat{\varphi}}\right)$ is upper-semicontinuous in p and attains its maximum at some $\hat{p}$. Further, there is a monotone sequence $\left(s^{(k)}\right)_{k=1}^{\infty}$ of realizations of signal S such that $\lim_{k\to\infty} \pi\left(p^{(k)}|G^{\hat{\varphi}}\right) = \pi\left(\hat{p}|G^{\hat{\varphi}}\right)$, *where $p^{(k)} \equiv \delta z^{(k)}$ and $z^{(k)} \equiv \mathbb{E}\left[\hat{\varphi}(X)|S = s^{(k)}\right]$.*

Going back to the original problem, let $z_i^{(k)} \equiv \mathbb{E}\left[\varphi_i(X)|S = s^{(k)}\right]$ and $p_i^{(k)} \equiv \delta z_i^{(k)}$. Note that

$$p^{(k)} = \delta\mathbb{E}\left[\hat{\varphi}(X)|S = s^{(k)}\right] = \delta\mathbb{E}\left[\sum_{i=1}^{I}\varphi_i(X)\middle|S = s^{(k)}\right] = \sum_{i=1}^{I} p_i^{(k)}.$$

The liquidity supplier i can guarantee herself at least $\pi\left(p_i^{(k)}\middle|G^{\varphi_i}\right)$ by posting a price $p_i^{(k)}$. Thus, the owner's payoff from selling $\boldsymbol{\varphi}$ under $\boldsymbol{G^{\varphi}}$ is bounded from above by

$$\sum_{i=1}^{I} V(G^{\varphi_i}) \leq \underbrace{\mu^{\hat{\varphi}}(1-\delta)}_{\text{maximal gains from trade}} - \underbrace{\sup_k \sum_{i=1}^{I} \pi\left(p_i^{(k)}\middle|G^{\varphi_i}\right)}_{\text{liquidity suppliers' profit guarantee}}.$$

Since $[X|S]$ is FOSD-ordered and φ_i is weakly increasing, $\mathbb{E}[\varphi_i(X)|S = s]$ is weakly increasing in s, and so, an owner type accepts $p_i^{(k)}$ for φ_i if, and only if, $s \leq s^{(k)}$. Similarly, in the auxiliary problem, an owner type accepts $p^{(k)}$ for $\hat{\varphi}$ if and only if $s \leq s^{(k)}$. We thus have

$$\begin{aligned}
\sum_{i=1}^{I} \pi\left(p_i^{(k)} \middle| G^{\varphi_i}\right) &= \sum_{i=1}^{I} \int_{-\infty}^{p_i^{(k)}/\delta} \left(z_i - p_i^{(k)}\right) \mathrm{d}G^{\varphi_i}\left(z_i\right) \\
&= \sum_{i=1}^{I} \int_{s \leq s^{(k)}} \left(\mathbb{E}\left[\varphi_i(X) | S = s\right] - p_i^{(k)}\right) \mathrm{d}G^{X}\left(s\right) \\
&= \int_{s \leq s^{(k)}} \left(\sum_{i=1}^{I} \mathbb{E}\left[\varphi_i(X) | S = s\right] - \sum_{i=1}^{I} p_i^{(k)}\right) \mathrm{d}G^{X}\left(s\right) \\
&= \int_{s \leq s^{(k)}} \left(\mathbb{E}\left[\hat{\varphi}(X) | S = s\right] - p^{(k)}\right) \mathrm{d}G^{X}\left(s\right) \\
&= \int_{-\infty}^{p^{(k)}/\delta} \left(z - p^{(k)}\right) \mathrm{d}G^{\hat{\varphi}}(z) = \pi\left(p^{(k)} \middle| G^{\hat{\varphi}}\right).
\end{aligned}$$

Thus,

$$\sum_{i=1}^{I} V\left(G^{\varphi_i}\right) \leq (1-\delta)\, \mu^{\hat{\varphi}} - \sup_{k} \pi\left(p^{(k)} \middle| G^{\hat{\varphi}}\right) = (1-\delta)\, \mu^{\hat{\varphi}} - \pi\left(\hat{p} | G^{\hat{\varphi}}\right).$$

By Theorem 1 and 2 in Kartik and Zhong (2023), an optimal signal S for security $\hat{\varphi}$ guarantees the payoff of $(1-\delta)\, \mu^{\hat{\varphi}} - \min_{\tilde{G}^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} \pi\left(P\left(\tilde{G}^{\hat{\varphi}}\right) \middle| \tilde{G}^{\hat{\varphi}}\right) \geq (1-\delta)\, \mu^{\hat{\varphi}} - \pi\left(\hat{p} \middle| G^{\hat{\varphi}}\right)$, which proves (14). □

Proof of Proposition 2. By Lemma 1, the information design program for $\tilde{\varphi}$ boils down to $\max_{\tilde{u} \leq \alpha \overline{x}} \left\{\tilde{u} : \mathcal{L}\left(\tilde{y} | \alpha\varphi, \tilde{u}\right) \geq 0, \tilde{y} \in [0, \tilde{u}]\right\}$. Using the fact that $H^{\alpha\varphi}(f) = H^{\varphi}\left(f/\alpha\right)$, $\mathcal{L}\left(\tilde{y} | \alpha\varphi, \tilde{u}\right) = \alpha\mu^{\varphi} - \delta\tilde{u} - (1-\delta)\, \tilde{y}^{1/(1-\delta)} \tilde{u}^{-\delta/(1-\delta)} + \int_{-\infty}^{\tilde{y}} H^{\varphi}\left(f/\alpha\right) \mathrm{d}f$. Consider $\tilde{u} = \alpha u$ and $\tilde{y} = \alpha y$. The information design problem reduces to solving

$$\max_{u \in [\underline{x}, \overline{x}]} \alpha\delta\left(u - \mu^{\varphi}\right) \;\; s.t. \;\; \alpha\left(\mu^{\varphi} - \delta u\right) \geq (1-\delta)\, \alpha y^{1/(1-\delta)} u^{-\delta/(1-\delta)} - \int_{-\infty}^{\alpha y} H^{\varphi}(f/\alpha) \mathrm{d}f, y \in [\underline{x}, u).$$

Doing a change of variables $t = f/\alpha$, the problem becomes

$$\max_{u \in [\underline{x}, \overline{x}]} \alpha\delta\left(u - \mu^{\varphi}\right) \;\; s.t. \;\; \mu^{\varphi} - \delta u \geq (1-\delta)\, y^{1/(1-\delta)} u^{-\delta/(1-\delta)} - \int_{-\infty}^{y} H^{\varphi}(t) \mathrm{d}t, y \in [\underline{x}, u),$$

thereby proving that any signal distribution that is optimal for $\tilde{\varphi}$ is also optimal for $\alpha\varphi$. □

Proof of Theorem 2. Fix any $(R_i)_{i=1}^{I}$ and let $R \equiv \sum_i R_i$. By Lemma 3, it is without loss of optimality to restrict attention to collection of securities $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^{I}$ satisfying $P\left(G^{\varphi_i}\right) \leq R_i$, for all $i = \overline{1, I}$. By Theorem 1, for any collection of securities $\boldsymbol{\varphi} = (\varphi_i)_{i=1}^{I}$, and any signal S about X inducing some $\boldsymbol{G^{\varphi}} \in \boldsymbol{\mathcal{G}^{\varphi}}$, $\sum_{i=1}^{I} V\left(G^{\varphi_i}\right) \leq \max_{G^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} V\left(G^{\hat{\varphi}}\right)$, where $\hat{\varphi}\left(X\right) \equiv \sum_{i=1}^{I} \varphi_i\left(X\right)$. By Proposition 1, $\max_{G^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} V\left(G^{\hat{\varphi}}\right) \leq V\left(G^{R}\right)$, where G^{R} is an optimal signal distribution for φ^{R} solving program (9). Thus, $V\left(G^{R}\right)$ is an upper bound for the security design problem (7). Let $\tilde{S}$ be a signal about X inducing G^{R} and let $u^{R} \equiv \sup\{\mathrm{supp} G^{R}\}$ be the highest expected security payoff induced by G^{R}. By Lemma 1 and the fact that $\left(\varphi^{R}, G^{R}\right)$ satisfy the constraint in 10,

$P\left(G^R\right) = \delta u^R \leq R$. Construct $(\alpha_i)_{i=1}^I$ so that $\delta\left(\alpha_i u^R\right) = R_i$. Define $\tilde{\varphi}_i(X) \equiv \alpha_i \varphi^R(X)$ and let $\boldsymbol{G}^{\tilde{\boldsymbol{\varphi}}}$ be the signal distributions of the security payoffs $\tilde{\boldsymbol{\varphi}} = (\tilde{\varphi}_i)_{i=1}^I$ induced by $\tilde{S}$. By Proposition 2, $\sum_{i=1}^I V\left(G^{\tilde{\varphi}_i}\right) = V\left(G^R\right)$. □

Proof of Proposition 3. Suppose that $\mathbb{P}\left[\sum_{i=1}^I \varphi_i(X) \neq X\right] > 0$. We argue below that $\boldsymbol{\varphi}$ is strictly suboptimal for program (7). Let $\boldsymbol{G}^{\boldsymbol{\varphi}} = (G^{\varphi_1}, \ldots, G^{\varphi_I})$ be an optimal signal for $\boldsymbol{\varphi}$ that solves program (6), and let S be the signal about X inducing $\boldsymbol{G}^{\boldsymbol{\varphi}}$. Further, suppose that $\boldsymbol{\varphi}$ satisfies that $P\left(G^{\varphi_i}\right) \leq R_i, i = \overline{1, I}$. If this is not the case, then by Lemma 3, there is a sequence of securities $\tilde{\boldsymbol{\varphi}} = (\tilde{\varphi}_i)_{i=1}^I$ constructed from $\boldsymbol{\varphi}$ satisfying these constraints that strictly dominates $\boldsymbol{\varphi}$.[13]

Next, define $\hat{\varphi}(X) \equiv \sum_{i=1}^I \varphi_i(X)$ and let $G^{\hat{\varphi}}$ be an optimal signal for $\hat{\varphi}$ that solves the program (9). By Theorem 1, $\bar{V}(\boldsymbol{\varphi}) \leq \max_{G^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} V\left(G^{\hat{\varphi}}\right)$. Next, consider the levered equity security $\varphi^R(X) \equiv \max\left\{X - D^R, 0\right\}$, $D^R \geq 0$, solving the program (10), and let G^R be an optimal signal distribution for φ^R solving program (9). Note that the assumption that $\mathbb{P}\left[\sum_{i=1}^I \varphi_i(X) \neq X\right] > 0$ implies that $\hat{\varphi}$ is not levered equity. In particular, $\mathbb{P}\left[\hat{\varphi}(X) \neq \varphi^R(X)\right] > 0$. Then, by Proposition 1, $\max_{G^{\hat{\varphi}} \in \mathcal{G}^{\hat{\varphi}}} V\left(G^{\hat{\varphi}}\right) \leq V\left(G^R\right)$, with strict inequality whenever $\mu^R - \delta u^R > 0$.

Finally, let $\boldsymbol{\varphi}^{\boldsymbol{R}} = \left(\varphi_i^{\boldsymbol{R}}\right)_{i=1}^I$ be the collection of common equity securities with $\varphi_i^{\boldsymbol{R}}(X) \equiv (R_i/R)\,\varphi^R(X), i = \overline{1, I}$, and $\boldsymbol{G}^{\boldsymbol{R}} \equiv \left(G_i^{\boldsymbol{R}}\right)_{i=1}^I$ be given by $G_i^{\boldsymbol{R}}(z) = G^R(Rz/R_i), i = \overline{1, I}$. By Theorem 2, $V\left(G^R\right) = \bar{V}\left(\boldsymbol{\varphi}^R\right)$. We conclude that $\bar{V}(\boldsymbol{\varphi}) \leq \bar{V}\left(\boldsymbol{\varphi}^R\right)$, with strict inequality whenever $\mu^R - \delta u^R > 0$. □

Proof of Corollary **??**. By Theorem 2, common equity attains the payoff of $\varphi^R(X)$, which is the maximal payoff attainable in (7) (by Theorem 1). Suppose there is another solution $\boldsymbol{\varphi}$ such that $\sum_{i=1}^I \varphi_i(X) \neq \varphi^R(X)$ with positive probability. Then, $\hat{\varphi}(X) \equiv \sum_{i=1}^I \varphi_i(X) \in \Phi$ and solves the security design problem for a single shock, which contradicts unique optimality of warrant in Theorem **??**. □

Lemma 5. *Consider securities $\varphi, \varphi_1, \varphi_2 \in \Phi_2$ such that $\varphi_1(X) + \varphi_2(X) = \varphi(X)$ almost surely and $u^{\varphi_1} + u^{\varphi_2} = u^{\varphi}$. Let $\varphi_i^e(X) \equiv (u^{\varphi_i}/u^{\varphi})\,\varphi(X), i = 1, 2$. Then,*

$$\sum_{i=1}^{2} \mathcal{L}\left(\varphi_i^e(\tilde{x}) | \varphi_i^e, u^{\varphi_i}\right) \geq \sum_{i=1}^{2} \mathcal{L}\left(\varphi_i(\tilde{x}) | \varphi_i, u^{\varphi_i}\right), \tilde{x} \in [\underline{x}, \overline{x}], \tag{18}$$

with a strict inequality whenever $\varphi_1(\tilde{x}) \neq \varphi_1^e(\tilde{x})$.

[13] The sequence $\tilde{\boldsymbol{\varphi}}$ is constructed as follows: for any $i = \overline{1, I}$ such that $P\left(G^{\varphi_i}\right) \leq R_i$, $\tilde{\varphi}_i(X) \equiv \varphi_i(X)$, whereas for any $i = \overline{1, I}$ such that $P\left(G^{\varphi_i}\right) > R_i$, $\tilde{\varphi}_i(X) \equiv \left(R_i/P\left(G^{\varphi_i}\right)\right)\varphi_i(X)$. The arguments below then should be applied to $\tilde{\boldsymbol{\varphi}}$.

Proof of Lemma 5. We have

$$\sum_{i=1}^{2}\mathcal{L}\left(\varphi_i\left(\tilde{x}\right)|\varphi_i,u^{\varphi_i}\right)=\mu^{\varphi}-\delta u^{\varphi}+\int_{-\infty}^{\tilde{x}}H(x)\dot{\varphi}\left(x\right)\mathrm{d}x-(1-\delta)\Big[\left(\varphi_1(\tilde{x})\right)^{1/(1-\delta)}\left(u^{\varphi_1}\right)^{-\delta/(1-\delta)}+\left(\varphi\left(\tilde{x}\right)-\varphi_1(\tilde{x})\right)^{1/(1-}$$

$$\leq\mu^{\varphi}-\delta u^{\varphi}+\int_{-\infty}^{\tilde{x}}H(x)\dot{\varphi}\left(x\right)\mathrm{d}x-(1-\delta)\Bigg[\left(\frac{u^{\varphi_1}\varphi\left(\tilde{x}\right)}{u^{\varphi_1}+u^{\varphi_2}}\right)^{1/(1-\delta)}\left(u^{\varphi_1}\right)^{-\delta/(1-\delta)}+\Bigg(\frac{u^{\varphi_2}\varphi\left(\tilde{x}\right)}{u^{\varphi_1}+u^{\varphi_2}}$$

$$=\sum_{i=1}^{2}\mathcal{L}\left(\varphi_i^e\left(\tilde{x}\right)|\varphi_i^e,u^{\varphi_i}\right)$$

where the inequality is by maximizing the right-hand side with respect to $\varphi_1\left(\tilde{x}\right)$. The inequality is strict whenever $\varphi_1\left(\tilde{x}\right)\neq\varphi_1^e\left(\tilde{x}\right)$. □

Proof of Proposition 4. Consider any collection $\boldsymbol{\varphi}=\left(\varphi_i\right)_{i=1}^{I}$ solving (7). By Corollary 1, $\sum_{i=1}^{I}\varphi_i\left(X\right)=\varphi^R\left(X\right)$. Denote $u^R\equiv u^{\varphi^R}$. By Lemma 3, since φ^R solves (10), $P\left(G^{\varphi^R}\right)=\delta u^R\leq R$. We first prove the result for $I=2$. We show in the proof of Proposition 2 in Inostroza and Tsoy (2022) that, when $\mathcal{X}=\{\underline{x},\overline{x}\}$, for any security φ, the information design program (9) boils down to

$$\max_{u\in[\varphi(\underline{x}),\varphi(\overline{x})]}u\ s.t.\ \mathcal{L}\left(\varphi\left(\underline{x}\right)|\varphi,u\right)=\mu^{\varphi}-\delta u-(1-\delta)\varphi\left(\underline{x}\right)^{1/(1-\delta)}u^{-\delta/(1-\delta)}\geq 0.$$

By $R<\delta\varphi^R\left(\overline{x}\right)$, $u^R<\varphi^R\left(\overline{x}\right)$, and so, $\mathcal{L}\left(\varphi^R\left(\underline{x}\right)\middle|\varphi^R,u^R\right)=0$. This also implies $\mathcal{L}\left(\varphi_i^e\left(\underline{x}\right)\middle|\varphi_i^e,u^{\varphi_i^e}\right)=0,i=1,2$. By Lemma 5, if $\varphi_1\left(\underline{x}\right)\neq\varphi_1^e\left(\underline{x}\right)$, then

$$0\leq\sum_{i=1}^{2}\mathcal{L}\left(\varphi_i\left(\underline{x}\right)|\varphi_i,u^{\varphi_i}\right)<\sum_{i=1}^{2}\mathcal{L}\left(\varphi_i^e\left(\underline{x}\right)\middle|\varphi_i^e,u^{\varphi_i^e}\right)=0,$$

which is not possible. Thus, $\varphi_1\left(\underline{x}\right)=\varphi_1^e\left(\underline{x}\right)$ and $\mathcal{L}\left(\varphi_i\left(\underline{x}\right)|\varphi_i,u^{\varphi_i}\right)=0,i=1,2$, which implies $\mu^{\varphi_i}=\delta u^{\varphi_i}-(1-\delta)\left(\varphi_i\left(\underline{x}\right)\right)^{1/(1-\delta)}\left(u^{\varphi_i}\right)^{-\delta/(1-\delta)}=\mu^{\varphi_i^e}$. Since $\mathcal{X}$ consists of two points, $\varphi_1\left(\overline{x}\right)=\varphi_1^e\left(\overline{x}\right)$, and so, φ_i coincides with φ_i^e, $i=1,2$.

By Theorem 1, $V\left(G^{\varphi_1}\right)+\sum_{i=2}^{I}V\left(G^{\varphi_i}\right)\leq V\left(G^{\varphi_1}\right)+\max_{G^{\hat{\varphi}_{-1}}\in\mathcal{G}^{\hat{\varphi}_{-1}}}V\left(G^{\hat{\varphi}_{-1}}\right)$, where $\hat{\varphi}_{-1}\left(X\right)\equiv\sum_{i=2}^{I}\hat{\varphi}_i\left(X\right)$. Thus, the owner's payoff is bounded from above by the payoff from choosing securities φ_1 and $\hat{\varphi}_{-1}$. The result then follows from the case of $I=2$. □

Proof of Proposition 5. Consider a relaxation of the owner's problem where she can choose a different signal S_i for each security $\varphi_i\left(X\right)$ rather than a single signal S about X for both of them. Fix any collection of securities $\boldsymbol{\varphi}=(\varphi_1,\varphi_2)\in\Phi$ and let $G^{\varphi_2}\in\mathcal{G}^{\varphi_2}$ be an arbitrary signal distribution. Because at round 2 the liquidity supplier faces a competitive liquidity supplier, the price offered by the latter equals $P_2=\mu^{\varphi_2}$, regardless of the distribution G^{φ_2}. The owner's relaxed problem can then be stated as follows

$$\sup_{(\varphi_1,\varphi_2)\in\boldsymbol{\Phi},G^{\varphi_1}\in\mathcal{G}^{\varphi_1}}V\left(G^{\varphi_1}\right)+(1-\delta)\mu^{\varphi_2}\text{ s.t. }P\left(G^{\varphi_1}\right)\leq R_1,\mu^{\varphi_2}\leq R_2.$$

Next, note that the owner's payoff is bounded from above by $\bar{V}\left(\varphi^{R_1}\right)$, where φ^{R_1} is the levered equity solving program (10) characterized in Proposition 1. The requirement that $\boldsymbol{\varphi} \in \Phi$ then implies that the owner can sell $\varphi_1(X) = \varphi^{R_1}(X)$ as long as $\varphi_2'(X) \in \left[0, 1 - \left(\varphi^{R_1}\right)'(X)\right]$ with probability 1. The owner then optimally sells the collection of securities $\boldsymbol{\varphi} = \left(\varphi^{R_1}, \varphi_2(X) \equiv \min\{X, d\}\right)$ with d chosen such that $\mu^{\varphi_2} = R_2$. $\square$

Online Appendix (Not for Publication)

Proof of Claim 1. Using the integration by parts formula for general distribution $G^{\hat{\varphi}}$ in Theorem VI.90 in Dellacherie and Meyer (1982),

$$\begin{aligned}\pi\left(\delta z|G^{\hat{\varphi}}\right) &= (1-\delta)\int_{\varphi(\underline{x})}^{z}\tilde{z}\mathrm{d}G^{\hat{\varphi}}\left(\tilde{z}\right)+\delta\int_{\varphi(\underline{x})}^{z}\left(\tilde{z}-\delta z\right)\mathrm{d}G^{\hat{\varphi}}\left(\tilde{z}\right)\\ &= (1-\delta)\int_{\varphi(\underline{x})}^{z}\tilde{z}\mathrm{d}G^{\varphi}\left(\tilde{z}\right)-\delta\int_{\varphi(\underline{x})}^{z}G^{\varphi}\left(\tilde{z}\right)\mathrm{d}\tilde{z}.\end{aligned}$$

Hence, $\pi\left(\delta z|G^{\hat{\varphi}}\right)$ is upper-semicontinuous in z, and so, it attains its maximum at some $\hat{z}$. Let $\hat{p}\equiv\delta\hat{z}$ be the corresponding optimal posted price.

By the upper-semicontinuity of $\pi\left(\delta z|G^{\hat{\varphi}}\right)$, either $\pi\left(\delta z|G^{\hat{\varphi}}\right)$ has an upward jump at $\hat{z}$ or it is continuous at $\hat{z}$. In the former case, the jump in $\pi\left(\delta z|G^{\varphi}\right)$ is due to a jump in $G^{\hat{\varphi}}$, and so, there is $\hat{s}$ such that $\mathbb{E}\left[\hat{\varphi}(X)|S=\hat{s}\right]=\hat{z}$ and $S=\hat{s}$ with positive probability. Thus, $s^{(k)}=\hat{s}$ for all k is the desired sequence.

To consider the latter case, suppose that $G^{\hat{\varphi}}$ assigns zero probability to a certain interval $\hat{Z}$, and let $(\hat{z},z']$ be the largest of such intervals (containing $\hat{Z}$). Then, there is $\varepsilon>0$ such that $G^{\hat{\varphi}}$ is strictly increasing over $(\hat{z}-\varepsilon,\hat{z})$. Hence, the liquidity supplier strictly prefers $p=\delta\hat{z}$ to any $p=\delta z, z\in(\hat{z},z']$, because increasing price from $\delta\hat{z}$ to δz does not increase probability of trade, but strictly increases the payout to the owner. Thus, there is a weakly increasing sequence $z^{(k)}\to\hat{z}$ belonging to the support of $G^{\hat{\varphi}}$ such that $\lim_{k\to\infty}\pi\left(\delta z^{(k)}|G^{\varphi}\right)=\pi\left(\delta\hat{z}|G^{\varphi}\right)$. Since $[X|S]$ are FOSD-ordered and $\hat{\varphi}(x)$ is weakly increasing in x, $\mathbb{E}[\hat{\varphi}(X)|S=s]$ is weakly increasing in s. Thus, there is a weakly increasing $s^{(k)}$ such that $z^{(k)}=\mathbb{E}\left[\hat{\varphi}(X)|S=s^{(k)}\right]$, which is the desired conclusion. $\square$